FURTHER EXTENSIONS ON A STABILITY PROPERTY WITH SLOWLY VARYING INPUTS

Hyungbo Shim\textsuperscript{a1} and Nam H. Jo\textsuperscript{b2}

\textsuperscript{a}ASRI, Department of Electrical Engineering
Seoul National University, Seoul, Korea
\textsuperscript{b}Department of Electrical Engineering
Soongsil University, Seoul, Korea

\texttt{namhoon@gmail.com}

Dedicated to Hassan K. Khalil on his 60th birthday

Abstract. In this paper we present three extended results on the stability property with slowly varying inputs that has been studied in, e.g., Khalil and Kokotović (1991). The stability property has played an important role in control methodologies such as gain-scheduling controls or robust controls for systems having slowly varying parameters. We begin with a review of the first extension, which shows that the inputs may vary fastly, but once its ‘average’ is slowly varying, then the same stability property still holds. The second extension deals with the case when a (transcritical) bifurcation occurs so that the assumptions of the stability property are violated. We propose a Jump & Wait strategy which will still guarantee the stability property even under this case. Finally we prove that the Jump & Wait strategy can be employed not with the input itself but with the average of the input if the proposed structural condition is satisfied.

Keywords. Slowly varying input; Averaging theory; Transcritical bifurcation; Positive system; Passing through bifurcation

1 Introduction

In the early 1990s, Kelemen [8], Lawrence and Rugh [12], and Khalil and Kokotović [9] have presented a stability property of nonlinear systems with slowly varying inputs, which can now be found in a graduate textbook such as [10]. The result is stated as follows: consider a dynamic system given by

\[ \dot{x} = f(x, u), \]  

\footnote{Research supported by Korea Research Foundation Grant KRF-2008-314-D00160.}

\footnote{Research supported by New & Renewable Energy R&D program(2009T100100621) under the Ministry of Knowledge Economy, Republic of Korea (MKE).}
where $f(\cdot, \cdot)$ is continuously differentiable, $x \in \mathbb{R}^n$, and $u \in \Gamma \subset \mathbb{R}^m$ in which $\Gamma$ is a connected compact set. Suppose that, for each frozen (i.e., constant) input $u \in \Gamma$, there exists a corresponding isolated equilibrium $x^*(u)$ such that $f(x^*(u), u) = 0$ and $x^*(\cdot)$ is continuous and piecewise twice continuously differentiable. Then, the following theorem holds.

**Theorem 1.** Let $B_R(x) \subset \mathbb{R}^n$ be an open ball of radius $R$ centered at $x$. Suppose that there exist a locally Lipschitz function $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and a positive constant $R$ such that, for all $u \in \Gamma$ and all $x \in B_R(x^*(u))$,

$$\alpha_1(\|x - x^*(u)\|) \leq V(x, u) \leq \alpha_2(\|x - x^*(u)\|), \quad (2)$$

$$\frac{\partial V}{\partial x} f(x, u) \leq -\alpha_3(\|x - x^*(u)\|), \quad \text{almost everywhere}, \quad (3)$$

where $\alpha_i(\cdot), i = 1, 2, 3$, are class-$K$ functions. Then, given $\rho > 0$, there exist $\delta > 0$ and $\kappa > 0$ such that, if the input $u(\cdot)$ is locally Lipschitz and satisfies

$$\left\| \frac{du}{dt}(t) \right\| \leq \kappa, \quad \text{almost everywhere,} \quad (4)$$

then the solution $x(t)$ of (1) with $\|x(0) - x^*(u(0))\| \leq \delta$ implies

$$\|x(t) - x^*(u(t))\| \leq \rho, \quad \forall t \geq 0.$$

Theorem 1 has been playing a crucial role for the stability proof of gain-scheduling control [13,14]. In fact, the utility of Theorem 1 becomes eminent when we want to drive the state $x(t)$, which is initially located in a neighborhood of $x^*(\tilde{u}_1)$, into a neighborhood of $x^*(\tilde{u}_2)$ where $\tilde{u}_1$ and $\tilde{u}_2$ are two different vectors. That is, if there exists a curve $\Gamma$ that connects two points $\tilde{u}_1$ and $\tilde{u}_2$ and the assumption of Theorem 1 holds, then the state $x(t)$ that is close to the point $x^*(\tilde{u}_1)$ can be driven into a small neighborhood of $x^*(\tilde{u}_2)$ by changing $u(t)$ sufficiently slowly from $\tilde{u}_1$ to $\tilde{u}_2$ along the curve $\Gamma$.

Two ingredients of the assumptions in Theorem 1 are (i) uniform stability in the sense that the inequalities (2) and (3) hold for all $x \in B_R(x^*(u))$ with a fixed $R$ independent of $u$, and the fact that (ii) the input $u(t)$ should be slowly varying. In this paper we relax those two conditions. In particular, Section 2 reviews the claim that the slowly varying average of $u(t)$ (not the slowly varying $u(t)$ itself) is enough for having the same result, which is a relaxation of the property (ii). In order to relax the property (i), Section 3 considers the case where the stability is lost due to the (transcritical) bifurcation that

---

3By the Rademacher’s theorem, a locally Lipschitz function has its derivative almost everywhere.

4As a matter of fact, local uniform asymptotic stability of $x^*(u)$, uniform in the parameter $u$, has been assumed in [9], and it was addressed that, by [5, Lemma 2], a Lyapunov function $V$ for (2) and (3) exists. Note that, a sufficient condition for the assumption is local exponential stability of $x^*(u)$ for every $u \in \Gamma$, which is easy to verify [10, Lemma 9.8].
may occur under the variation of \( u \). Section 4 presents the combined result of Sections 2 and 3. Section 5 concludes the paper.

**Notation:** We denote by \( e_k \) the column vector \([0 \cdots 0 1 0 \cdots 0]^T\) with the entry 1 in the \( k \)-th place. The elementary matrix obtained by interchanging the first and the \( k \)-th row of the identity matrix is denoted by \( E_k \). When all eigenvalues of a matrix \( A \) has negative real parts, \( A \) is called a *Hurwitz* matrix. The maximum and the minimum eigenvalues of a symmetric matrix \( P \) are denoted by \( \lambda_{\text{max}}(P) \) and \( \lambda_{\text{min}}(P) \), respectively. For a set \( M \), the closure and the boundary of \( M \) are denoted by \( \bar{M} \) and \( \partial M \), respectively. Let \( \mathbb{R}^n_+ := \{ x \in \mathbb{R}^n : x_1 > 0, \cdots, x_n > 0 \} \) and \( B_R(x_0) := \{ x \in \mathbb{R}^n : \| x - x_0 \| < R \} \) for some \( R > 0 \) and \( x_0 \in \mathbb{R}^n \). The notation \( x \in (a, b) \) implies that \( a < x < b \) while \( x \in [a, b] \) is for \( a \leq x \leq b \).

## 2 Stability Property with Inputs having Slowly Varying Average

In this section, we consider an input that may not be slowly varying itself, but whose average is slowly varying. Suppose that the system is given by

\[
\dot{x}(t) = f(x(t), u_a(t), u_f(\frac{1}{\epsilon} t)), \quad u_a \in \mathbb{R}^m, \quad u_f \in \mathbb{R}^l, \quad (5)
\]

where \( f(\cdot, \cdot, \cdot) \) is continuously differentiable, \( u_a(\cdot) \) is locally Lipschitz, and \( u_f(\cdot) \) is a uniformly bounded measurable function that is periodic with a period \( T > 0 \). The small positive parameter \( \epsilon \) (to be specified) indicates that the function \( u_f \) varies fastly, relatively to the behavior of the system state \( x(t) \). In fact, we are going to rely on three-time scale behavior of the system by specifying \( \epsilon \) sufficiently small so that \( u_f \) oscillates sufficiently fast and by specifying the upper bound of the derivative of \( u_a \) (i.e., \( \| \frac{du_a}{dt} \| \leq \kappa \) almost everywhere with sufficiently small \( \kappa \)) so that \( u_a \) varies sufficiently slowly.

**Remark 1.** A system having an input \( u(t) \) whose average is slowly varying can be seen as

\[
\dot{x}(t) = f(x(t), u(t)) = f(x(t), u_a(t) + u_f(\frac{1}{\epsilon} t))
\]

where \( u_a(t) \) is slowly varying and the function \( u_f(\cdot) \) has zero mean. In this way, this system is cast into (5).

For now, the argument of \( u_f \), i.e. \( t/\epsilon \), is treated as an independent time \( \tau \), and the system (5) is written as

\[
\dot{x}(t) = f_{av}(x(t), u_a(t)) + f_p(x(t), u_a(t), u_f(\tau)) \quad (6)
\]

where \( f_{av} \) is defined as

\[
f_{av}(x, u_a) := \frac{1}{T} \int_0^T f(x, u_a, u_f(s)) ds
\]
and $f_p(x, u_a, u_f) := f(x, u_a, u_f) - f_{av}(x, u_a)$. Note that $\int_0^T f_p(x, u_a, u_f(s))ds = 0$ for any $x$ and $u_a$.

Now we formally assume stability of the averaged system with frozen input $u_a$ as follows.

**Assumption 1.** The averaged system

$$\dot{x} = f_{av}(x, u_a)$$

(7)

has an isolated equilibrium $x^*(u_a)$ for each frozen input $u_a \in \Gamma$, that is, $f_{av}(x^*(u_a), u_a) = 0$, and $x^*(\cdot)$ is continuous and piecewise twice continuously differentiable. In addition, there exist a locally Lipschitz function $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and a positive constant $R$ such that, for each $u_a \in \Gamma$ and each $x \in B_R(x^*(u_a))$,

$$\alpha_1(||x - x^*(u_a)||) \leq V(x, u_a) \leq \alpha_2(||x - x^*(u_a)||)$$

(8)

$$\frac{\partial V}{\partial x} f_{av}(x, u_a) \leq -\alpha_3(||x - x^*(u_a)||), \text{ a.e.}$$

(9)

where $\alpha_i(\cdot), i = 1, 2, 3$, are class-$K$ functions.

Then, the following theorem is an extension of Theorem 1, whose proof is found in [3], and also included in the Appendix for convenience.

**Theorem 2.** Consider the system (5) under Assumption 1. Given $\rho > 0$, there exist positive constants $\epsilon^*$, $\kappa$ and $\delta$ such that, if $0 < \epsilon < \epsilon^*$, and $u_a(\cdot)$ is locally Lipschitz and satisfies

$$\left\| \frac{du_a}{dt}(t) \right\| \leq \kappa, \text{ a.e.},$$

(10)

then the solution $x(t)$ of (5) with $\|x(0) - x^*(u_a(0))\| \leq \delta$ implies

$$\|x(t) - x^*(u_a(t))\| \leq \rho, \quad \forall t \geq 0.$$

### 3 Passing Through Bifurcated Equilibrium by Jump-and-Wait Strategy

One apparent case that the uniform stability assumption in Theorem 1 breaks down is when a bifurcation occurs so that more than two equilibrium points meet together at one point in the state-space. In particular, we consider the following nonlinear system

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, u \in \Gamma \subset \mathbb{R},$$

(11)

where $\Gamma = [\bar{u}_1, \bar{u}_2]$. We assume that the system is a positive system\(^5\) for each $u \in \Gamma$. Also suppose that there exist $C^3$ functions $x^A : \Gamma \to \mathbb{R}^n_+$ and $x^B :$

\(^5\)A positive system implies that all the states remain in the positive orthant (i.e., the subset of the state-space in which all the states are positive) when the initial state is there. A nonnegative system can be similarly defined, but it has been shown that they are equivalent if the system is locally Lipschitz [1, p. 214].
Extensions on a Stability Property

\[ \Gamma \to \mathbb{R}_+^n \] such that \( f(x^A(u), u) = f(x^B(u), u) = 0 \) for all \( u \in \Gamma \). In addition, for some \( u^* \) with \( \bar{u}_1 < u^* < \bar{u}_2 \), we assume that \( x^A(u^*) = x^B(u^*) \in \partial \mathbb{R}_+^n \) and an exchange of stability occurs at \( u = u^* \), i.e., \( \frac{\partial^2 f}{\partial u^2}(x^A(u), u) \) (respectively, \( \frac{\partial^2 f}{\partial u^2}(x^B(u), u) \)) is Hurwitz only for \( u \in [\bar{u}_1, u^*) \) (respectively, \( u \in (u^*, \bar{u}_2) \)). We let the equilibrium curve \( x^*(u) \) be

\[
x^*(u) := \begin{cases} x^A(u), & u \in [\bar{u}_1, u^*] \\ x^B(u), & u \in [u^*, \bar{u}_2]. \end{cases}
\]

Now, suppose that our control objective is, for given \( \rho > 0 \), to drive the state \( x(t) \), which is initially located in a neighborhood of \( x^A(\bar{u}_1) \), to a neighborhood of \( x^B(\bar{u}_2) \) while maintaining \( \|x(t) - x^*(u(t))\| < \rho \). By the construction, for \( u \in [\bar{u}_1, u^*] \cup (u^*, \bar{u}_2) \), \( x^*(u) \) is asymptotically stable since \( \frac{\partial^2 f}{\partial u^2}(x^*(u), u) \) is Hurwitz. On the other hand, if we restrict our interest to the positive orthant, then it may happen that \( x^*(u) \) is asymptotically stable (with respect to the positive orthant) for all frozen \( u \in [\bar{u}_1, \bar{u}_2] \), and its basin of attraction does not shrink to zero.\(^6\) In such a case, one may expect from Theorem 1 that our control goal can be achieved by applying a slowly increasing control, e.g., \( u(t) = \bar{u}_1 + \kappa t \) with a sufficiently small constant \( \kappa > 0 \). However, this approach may not work since Theorem 1 requires not only asymptotic stability of \( x^*(u) \) but also the existence of an appropriate Lyapunov function \( V(x) \) that satisfies (2) and (3) (with a class-\( \mathcal{K} \) function \( \alpha_3 \) in (3)).

To illustrate this point further, consider the following simple system

\[
\begin{align*}
\dot{x} &= u - x - uxy \\
\dot{y} &= uxy - y
\end{align*}
\]

(12)

where it is assumed that \( \bar{u}_1 \leq u \leq \bar{u}_2 \) with \( \bar{u}_1 = 0.9 \) and \( \bar{u}_2 = 1.1 \). It is easily checked by the essential nonnegative condition \([2]\) that the set \( \mathbb{R}_+^2 \) is forward invariant for every \( u \in [\bar{u}_1, \bar{u}_2] \), which implies that the system (12) is a positive system. The system (12) has two equilibria, \( x^A(u) = (u, 0) \) and \( x^B(u) = (1/u, u - 1/u) \). It is seen from the Jacobian linearization that \( x^A(u) \) is locally exponentially stable when \( u \in [\bar{u}_1, u^*) \) while \( x^B(u) \) is locally exponentially stable when \( u \in (u^*, \bar{u}_2) \). Stability of \( x^A(u^*) = x^B(u^*) \) is not determined since the Jacobian matrix at \( x^A(u) \) is given by

\[
\begin{bmatrix}
-1 & -u^2 \\
0 & u^2 - 1
\end{bmatrix},
\]

which has one eigenvalue at the origin when \( u = u^* \). Let the equilibrium

\[^6\text{This property has been termed as non-vanishing basin of attraction stability (with respect to the positive orthant) \([16]\). One may wonder whether there is a case where the non-vanishing basin of attraction stability does not hold. Indeed, the example in [6] shows this. Consider a system } \dot{x} = -x(x^2 - u^2)^2. \text{ Then, for each frozen } u \in [0, 1], \text{ the origin is locally asymptotically stable with the basin of attraction } \Omega_u := (-u, u) \text{ when } u \neq 0, \text{ and the whole interval } \Omega_0 := \mathbb{R} \text{ when } u = 0. \text{ However, the set } \cap_{u \in [0, 1]} \Omega_u \text{ is empty.}\]
the curve of our interest be
\[
x^*(u) = \begin{cases} (u, 0), & \text{when } \bar{u}_1 \leq u \leq u^* \\ \left(\frac{1}{u}, u - \frac{1}{u}\right), & \text{when } u^* \leq u \leq \bar{u}_2. \end{cases}
\] (13)

Now, for given \( \rho > 0 \), suppose that the control goal is to steer the state initially located in a neighborhood of \( x^A(\bar{u}_1) \) into a neighborhood of \( x^B(\bar{u}_2) \) while keeping \( \|x(t) - x^*(u(t))\| < \rho \) for all \( t \geq 0 \). Since \( x^*(u) \) is stable for all \( u \in [\bar{u}_1, u^*) \cup (u^*, \bar{u}_2] \), the only point where the stability is in doubt is at \( u = u^* \). Although equilibrium \( x^A(u^*) = x^B(u^*) \) is not stable in the usual sense, it can be shown by \cite{16} that \( x^*(u) \) is locally asymptotically stable for each \( u \in [\bar{u}_1, \bar{u}_2] \) if we restrict our concern into the positive orthant \( \mathbb{R}_+^2 \). Moreover, it can be shown that the size of the basin of attraction of \( x^*(u) \) (with respect to \( \mathbb{R}_+^2 \)) does not shrink to zero for all \( u \in [\bar{u}_1, \bar{u}_2] \). Nonetheless, Theorem 1 cannot be applied to system (12) because the uniform stability does not hold. In fact, suppose that there exists a function \( V(x, u) \) such that
\[
\frac{\partial V}{\partial x} f(x, u) < 0, \quad \text{for all } x \in B_R(x^*(u)) \cap \mathbb{R}_+^1, \ u \in [\bar{u}_1, \bar{u}_2],
\]
with some \( R > 0 \) and \( f = \begin{bmatrix} u - x - uxy, uxy - y \end{bmatrix}^T \). Then, there is no class-\( \mathcal{K} \) function \( \alpha_3 \) such that, for all \( x \in B_R(x^*(u)) \cap \mathbb{R}_+^1, \ u \in [\bar{u}_1, \bar{u}_2], \)
\[
\frac{\partial V}{\partial x} f(x, u) \leq -\alpha_3(\|x - x^*(u)\|).
\] (14)

To see this, suppose to the contrary that there exists a class-\( \mathcal{K} \) function \( \alpha_3 \) that satisfies (14). Pick \( u^*_+ > u^* \) such that \( x^A(u^*_+) \) is located in \( B_R(x^*(u^*_+)) \cap \mathbb{R}_+^2 \). Then, \( \frac{\partial V}{\partial x} f(x, u^*_+) \bigg|_{x=x^A(u^*_+)} = 0 \) but \( -\alpha_3(\|x - x^*(u^*_+)\|) \bigg|_{x=x^A(u^*_+)} < 0 \), which violates (14).

Now, we return to the general problem. To accomplish our goal without help of an appropriate Lyapunov function, we utilize the following assumption of \cite{16}.

**Assumption 2.** For some integer \( k \) \( (1 \leq k \leq n) \), the \( k \)-th element of \( x^*(u^*) \) is zero and the \( k \)-th row of the Jacobian \( \frac{\partial f}{\partial x}(x^*(u^*), u^*) \) is the zero vector. Moreover, \( (d^2\psi)/(ds^2)(0) < 0 \), where
\[
\psi(s) := E_k \begin{bmatrix} 1 \\ -\bar{A}_{21} \end{bmatrix} s + x^*(u^*), u^*
\]
in which \( \bar{A}_{21} \in \mathbb{R}^{(n-1) \times 1} \) and \( \bar{A}_{22} \in \mathbb{R}^{(n-1) \times (n-1)} \) are obtained by
\[
E_k \begin{bmatrix} f(x^*(u^*), u^*)E_k^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}.
\]

It has been shown in \cite{16} that Assumption 2 implies that \( x^*(u^*) \) is asymptotically stable with respect to \( \mathbb{R}_+^n \) and there exists \( R > 0 \), independent of
 Extensions on a Stability Property

Figure 1: Schematic of Jump & Wait strategy. The black solid (respectively, dashed) line indicates that equilibria along the corresponding equilibrium curve are stable (respectively, unstable). The shaded region represents the forward invariant set $R^n_+$. 

$u$, such that $B_R(x^*(u)) \cap R^n_+$ is contained in the basin of attraction of $x^*(u)$ for all $u \in \Gamma$. With such an $R$, pick positive $\sigma_-$, $\sigma_+$, and $\rho_-$ such that $B_{\rho_-}(x^*(u^* - \sigma_-)) \subset B_R(x^*(u^* + \sigma_+))$. (See Fig. 1.) Here, without loss of generality, we assume that $\bar{u}_1 < u^* - \sigma_-$ and $u^* + \sigma_+ < \bar{u}_2$.

Now, since $\frac{\partial f}{\partial x}(x^*(u), u)$ is Hurwitz for each $u \in [\bar{u}_1, u^* - \sigma_-]$, Theorem 1 implies that, for given $\rho_-(\leq \rho)$, there exist positive constants $\kappa_-$ and $\delta_-$ such that, with $\|du/dt(t)\| \leq \kappa_-$,

$$x(0) \in B_{\delta_-}(x^*(\bar{u}_1)) \cap R^n_+ \implies x(t) \in B_{\rho_-}(x^*(u(t))) \cap R^n_+$$

where $u(t) \in [\bar{u}_1, u^* - \sigma_-]$. Similarly, for given $\rho_+ \leq \rho$, there exist positive constants $\kappa_+$ and $\delta_+$ such that, with $\|du/dt(t)\| \leq \kappa_+$,

$$x(0) \in B_{\delta_+}(x^*(u^* + \sigma_+)) \cap R^n_+ \implies x(t) \in B_{\rho_+}(x^*(u(t))) \cap R^n_+$$

where $u(t) \in [u^* + \sigma_+, \bar{u}_2]$. Moreover, since $B_{\rho_-}(x^*(u^* - \sigma_-)) \cap R^n_+$ is included in the basin of attraction of $x^*(u^* + \sigma_+)$, there exists a time $T_{\text{wait}} > 0$ such that, with $u = u^* + \sigma_+$,

$$x(0) \in B_{\rho_-}(x^*(u^* - \sigma_-)) \cap R^n_+ \implies x(t) \in B_{\delta_+}(x^*(u^* + \sigma_+)) \cap R^n_+$$

for any $t > T_{\text{wait}}$. We summarize our discussion so far in the following algorithm, which will be referred to as ‘Jump & Wait strategy’ in the remainder
of the paper.7

Jump & Wait strategy:

**Step 1:** Monotonically increase $u(t)$ from $\bar{u}_1$ to $u^* - \sigma_-$ while maintaining $\|\frac{du}{dt}(t)\| \leq \kappa_-$. Let $t^*$ be the time when $u(t^*) = u^* - \sigma_-$. 

**Step 2:** Apply $u(t) = u^* + \sigma_+$ for $t^* \leq t \leq t^* + T_{\text{wait}}$ where $T_{\text{wait}}$ is a sufficiently large constant. (The minimum value of $T_{\text{wait}}$ generally depends on the state $x(t^*)$.)

**Step 3:** For $t \geq t^* + T_{\text{wait}}$, monotonically increase $u(t)$ from $u^* + \sigma_+$ to $\bar{u}_2$ while maintaining $\|\frac{du}{dt}(t)\| \leq \kappa_+$. 

Here we emphasize that just varying $u(t)$ sufficiently slowly is not enough to achieve the goal, which is illustrated by the following simplified example.

**Example.** Consider the positive system

$$\dot{x} = -x(x - u).$$

The system has two equilibria at $x = 0$ and $x = u$, and the bifurcation point is $u = 0 =: u^*$. Suppose that our interest is the equilibrium curve

$$x^*(u) = \begin{cases} 0, & u < 0 \\ u, & u \geq 0 \end{cases}$$

which satisfies all the assumptions in this section. Let $\bar{u}_1 = -0.1$ and $\bar{u}_2 = 0.3$, and let $u(t) = \kappa t + \bar{u}_1$, $\kappa > 0$. A computer simulation has been performed to investigate the system behavior as $\kappa$ gets smaller. Fig. 2 shows simulation results with $x(0) = 0.03$ and it is seen that $\|x(t) - x^*(u(t))\| < 0.03$ cannot be achieved no matter how smaller $\kappa$ is chosen. Therefore, it is clear that, just by changing $u(t)$ sufficiently slowly, $\|x(t) - x^*(u(t))\| < \rho$ is not attainable.

### 4 Passing Through Bifurcated Equilibrium with Inputs having Slowly Varying Average

Now we again consider the system (5), or equivalently the system (6):

$$\dot{x}(t) = f(x(t), u_a(t), u_f(t/\epsilon)) = f_{av}(x(t), u_a(t)) + f_p(x(t), u_a(t), u_f(t/\epsilon))$$

where $f_{av}(x, u_a) = (1/T) \int_0^T f(x, u_a, u_f(s))ds$ and $f_p(x, u_a, u_f) = f(x, u_a, u_f) - f_{av}(x, u_a)$. For convenience, we assume that $u_a \in \Gamma \subset \mathbb{R}$ where $\Gamma$ is a connected compact interval $[\bar{u}_1, \bar{u}_2]$, $u_f \in F \subset \mathbb{R}^I$ where $F$ is a compact set, and the system (16) is a positive system for every $u_a \in \Gamma$ and $u_f \in F$. Our concern here is to establish a result similar to Theorem 2 without relying

---

7A similar strategy has been used in [7].
Figure 2: The state trajectories of system (15) with $x(0) = 0.3$ and $u = \kappa t + \bar{u}_1$. The dash-dotted, the dashed, and the solid lines represent the results of $\kappa = 2 \cdot 10^{-3}$, $\kappa = 1 \cdot 10^{-3}$, and $\kappa = 2 \cdot 10^{-4}$, respectively. In addition, the dotted red line represents $x^*(u)$.

on a condition like Assumption 1. For this, we additionally assume that the $k$-th element $f_k$ of the vector $f$ has the following form

$$f_k(x, u_a, u_f) = x_k \tilde{f}_k(x, u_a, u_f)$$

where $\tilde{f}_k$ is continuously differentiable. (As a consequence, it also follows that $f_{av,k}(x, u_a) = x_k f_{av,k}(x, u_a)$ and $f_{p,k}(x, u_a, u_f) = x_k f_{p,k}(x, u_a, u_f)$.)

For the time being, we would discuss on the averaged system

$$\dot{x} = f_{av}(x, u_a),$$

which is a positive system for each $u_a \in \Gamma$ by construction. The system (18) is supposed to have two equilibria for each $u_a \in \Gamma$, which are denoted by $x^A(u_a)$ and $x^B(u_a)$ respectively. It is assumed that they are $C^3$ functions.

In addition, we assume that the transcritical bifurcation occurs when $u_a = u^* \in (\bar{u}_1, \bar{u}_2)$ so that they collide with each other at $u^*$ and exchange their stability. Let the curve of stable equilibria (stable except at $u^*$) be $x^*(u_a)$ and the curve of unstable equilibria (unstable except at $u^*$) be $x^\circ(u_a)$. When $u_a = u^*$, we assume that the Jacobian of $f_{av}$ at $x^*(u_a) = x^\circ(u_a)$ has only one eigenvalue at the origin and the rest $n-1$ eigenvalues are in the open left-half complex plane. Moreover, it is assumed that

$$x_k^*(u^*) = 0 \quad \text{and} \quad \tilde{f}_{av,k}(x^*(u^*), u^*) = 0.$$  

A consequence of this assumption is that the $k$-th row of the Jacobian of $f_{av}$ at $x^*(u^*)$ when $u_a = u^*$ is the zero vector. Hence, we are able to assume the following.
**Assumption 3.** Let \( \bar{A}_{21} \in \mathbb{R}^{(n-1)\times 1} \) and \( \bar{A}_{22} \in \mathbb{R}^{(n-1)\times (n-1)} \) be obtained by

\[
E_k \frac{\partial f_{av}}{\partial x}(x^*(u^*), u^*) E_k^{-1} =: \begin{bmatrix} 0 & 0 \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}.
\]

Then, the function \( \psi(s) \), defined by

\[
\psi(s) = f_{av,k} \left( E_k \begin{bmatrix} 1 \\ -\bar{A}_{22}^{-1} \bar{A}_{21} \end{bmatrix} s + x^*(u^*), u^* \right),
\]

satisfies the condition that \( (\partial \psi^2)/(\partial s^2)(0) < 0 \).\(^8\)

Now we suppose that the function \( u_f(\cdot) \) is given for the system (16). Our goal is to design \( u_a(\cdot) \) and choose a suitable \( \epsilon \) so that the solution \( x(t) \) of (16), initiated in a neighborhood of \( x^*(\bar{u}_1) \), is transferred to a neighborhood of \( x^*(\bar{u}_2) \) while remaining in the \( \rho \)-neighborhood of the curve \( x^* \) (where \( \rho \) is given a priori). The design is based on Theorem 2 and the Jump & Wait strategy.

Even with the fast varying \( u_f \), transferring the state \( x(t) \) along the curve \( x^*(\cdot) \) from \( \bar{u}_1 \) to \( u^* - \sigma_\epsilon \), and from \( u^* + \sigma_\epsilon \) to \( \bar{u}_2 \) (with any positive \( \sigma_- \) and \( \sigma_+ \)) is easy. Indeed, by Theorem 2, for any \( \rho_- > 0 \), there exist positive \( \epsilon_-, \delta_-, \) and \( \kappa_- \) such that \( x(t) \in B_{\rho_-}(x^*(u(t))) \cap \mathbb{R}^n_+ \) if \( x(0) \in B_{\delta_-}(x^*(\bar{u}_1)) \cap \mathbb{R}^n_+ \), \( \epsilon \leq \epsilon_- \), and \( \|\frac{dx}{dt}(t)\| \leq \kappa_- \). Likewise, for any \( \rho_+ > 0 \), there exist positive \( \epsilon_+, \delta_+, \) and \( \kappa_+ \) such that \( x(t) \in B_{\rho_+}(x^*(u(t))) \cap \mathbb{R}^n_+ \) if \( x(0) \in B_{\delta_+}(x^*(u^* + \sigma_+)) \cap \mathbb{R}^n_+ \), \( \epsilon \leq \epsilon_+ \), and \( \|\frac{dx}{dt}(t)\| \leq \kappa_+ \). Therefore, if we employ the Jump & Wait strategy of the previous section, the remaining question is whether the state in \( B_{\rho_-}(x^*(u^* - \sigma_-)) \cap \mathbb{R}^n_+ \) can be driven to \( B_{\delta_+}(x^*(u^* + \sigma_+)) \cap \mathbb{R}^n_+ \). During this period \( T_{\text{wait}} \), we will keep \( u_a = u^* + \sigma_+ \) and see whether the state \( x(t) \) is attracted by a neighborhood of \( x^*(u^* + \sigma_+) \) under the fluctuating signal \( u_f \).

We note here that, under the assumptions so far, there could be only two cases:

(Case 1) \( x_k^0(u_a) = 0 \) and \( x_k^0(u_a) > 0 \) for all \( u_a \) slightly larger than \( u^* \),

(Case 2) \( x_k^0(u_a) = 0 \) and \( x_k^0(u_a) < 0 \) for all \( u_a \) slightly larger than \( u^* \).

To see this, we first recall that the Jacobian of \( f_{av} \) at \( x^*(u^*) \) has only one eigenvalue at the origin and its \( k \)-th row is the zero vector. Since the other rows of the Jacobian are linearly independent, it follows from the implicit function theorem that the equilibrium point of \( f_{av}(x, u_a) \) is uniquely determined around \( u^* \) by its own \( k \)-th component. Since the equilibrium, when \( u_a = u^* \), is \( x^*(u^*) = x^0(u^*) \), whose \( k \)-th component is the same as zero,

\(^8\)This is equivalent to the condition that \( (\partial \psi^2)/(\partial s^2)(0) < 0 \) where

\[
\psi(s) = f_{av,k} \left( E_k \begin{bmatrix} 1 \\ -\bar{A}_{22}^{-1} \bar{A}_{21} \end{bmatrix} s + x^*(u^*), u^* \right).
\]
Extensions on a Stability Property

one of them should have different $k$-th component when $u_a$ becomes different from $u^*$ (otherwise $x^*(u_a) = u^0(u_a)$ even if $u_a \neq u^*$ which is not the case). The argument also guarantees that the $k$-th component of either $x^*(u_a)$ or $x^0(u_a)$ should be zero (because $f_{av,q}(x, u_a) = x_k f_{av,k}(x, u_a)$ and only two equilibrium points are there). In addition, it can be shown (by following the method in [16]) that $x^0(u_a) < x^*(u_a)$ for $u_a$ slightly larger than $u^*$. Therefore, it suffices to consider the two cases only.

We consider the Case 1 first. For now, let $u_a$ be a constant belonging to the interval $[u^*, \bar{u}_2]$. Let

$$A(u_a) = \frac{\partial f_{av}(x^0(u_a), u_a)}{\partial x}.$$  

Then, it can be shown (see the Appendix) that there exist $u_2^*$ such that $u^* < u_2^* \leq \bar{u}_2$, and a real invertible matrix function $T(u_a)$ such that the first row of $T(u_a)$ is $e_k^T$ for all $u_a \in [u^*, u_2^*]$ and

$$T(u_a)A(u_a)T^{-1}(u_a) = \begin{bmatrix} A_1(u_a) & 0 \\ 0 & A_2(u_a) \end{bmatrix}, \quad \forall u_a \in [u^*, u_2^*]$$  

(20)

where $A_1(u^*) = 0$, $A_1(u_a) > 0$ for $u_a \in (u^*, u_2^*]$, and $A_2(u_a) \in \mathbb{R}^{(n-1) \times (n-1)}$ is Hurwitz for all $u_a \in [u^*, u_2^*]$. Now, define

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = T(u_a)(x - x^0(u_a)),$$

from which $z = 0$ corresponds to $x^0(u_a)$. Then, (16) can be transformed into

$$\dot{z} = T(u_a)f_{av}(T^{-1}(u_a)z + x^0(u_a), u_a) + T(u_a)f_p(T^{-1}(u_a)z + x^0(u_a), u_a, u_f)$$

$$=: g_{av}(z, u_a) + g_p(z, u_a, u_f),$$  

(21)

in which, $\int_0^T g_p(z, u_a, u_f(z))ds = 0$ by construction.

Now we focus on the averaged system

$$\dot{z} = g_{av}(z, u_a).$$  

(22)

The Jacobian of (22) at $z = 0$ is the matrix in (20), and it can be shown that there exists an invariant manifold $\{(z_1, z_2) \in \mathcal{N} : z_2 = \pi(z_1, u_a)\}$ for each $u_a$ locally around $u^*$, where $\mathcal{N}$ is a neighborhood of the origin and the function $\pi$ satisfies that

$$\pi(0, u_a) = 0, \quad \frac{\partial \pi}{\partial z_1}(0, u^*) = 0,$$

(23)

and $g_{av,2}(z_1, \pi(z_1, u_a), u_a) = \frac{\partial \pi}{\partial z_1}g_{av,1}(z_1, \pi(z_1, u_a), u_a)$.

---

9When we mention ‘locally around $u^*$’, it means an interval $[u^*, u']$ where $u'$ is a certain constant such that $u^* < u' \leq u_2^*$, which we do not care much about because we can always take a smaller interval for the analysis.
(For the derivation, see the Appendix.) The first equation implies that the manifold includes the origin for all \( u_a \), and the second means that it is tangential to the \( z_1 \)-axis when \( u_a = u^* \). The invariance follows from the third. In fact, the manifold becomes the center manifold when \( u_a = u^* \).

Now, define \( w := z_2 - \pi(z_1, u_a) \). Then, we obtain

\[
\begin{align*}
\dot{z}_1 &= g_{av,1}(z_1, w + \pi(z_1, u_a), u_a) =: h_{av,1}(z_1, w, u_a) \\
\dot{w} &= g_{av,2}(z_1, w + \pi(z_1, u_a), u_a) - \frac{\partial \pi}{\partial z_1}(z_1, u_a) g_{av,1}(z_1, w + \pi(z_1, u_a), u_a) \\
&=: h_{av,2}(z_1, w, u_a)
\end{align*}
\]

(24)

in which, \( h_{av,2}(z_1, 0, u_a) = 0 \) by construction. The overall coordinate change is given by

\[
\begin{align*}
z_1 &= x_k \\
w &= T(u_a)[2, n](x - x^0(u_a)) - \pi(z_1, u_a)
\end{align*}
\]

(25)

where \( T(u_a)[2, n] \) means the second to \( n \)-th rows of \( T(u_a) \). Hence, the equilibrium \( x^0(u_a) \) corresponds to the origin of (24), and at the origin the Jacobian of the system (24) is given by

\[
\begin{bmatrix}
A_1(u_a) & 0 \\
0 & A_2(u_a)
\end{bmatrix}.
\]

Therefore, the \( w \)-dynamics of (24) can be written as

\[
\dot{w} = h_{av,2}(z_1, w, u_a) = A_2(u_a)w + N_2(z_1, w, u_a)
\]

(26)

where \( N_2(z_1, w, u_a) = h_{av,2}(z_1, w, u_a) - A_2(u_a)w \) satisfies that \( N_2(z_1, 0, u_a) = 0 \) and \( \partial N_2/\partial w(0, 0, u_a) = 0 \) because the linear term is \( A_2(u_a)w \) only in the equation. Likewise, the \( z_1 \)-dynamics of (24) can be written, under the property of (17), as

\[
\begin{align*}
\dot{z}_1 &= h_{av,1}(z_1, w, u_a) = z_1 h_{av,1}(z_1, w, u_a) = A_1(u_a)z_1 + z_1 R_1(z_1, w, u_a)
\end{align*}
\]

(27)

where \( R_1(0, 0, u_a) = 0 \).

Since \( A_2(u_a) \) is Hurwitz locally around \( u^* \), it is seen that the bifurcated equilibrium of \( x^0(u_a) \) (i.e., \( x^*(u_a) \)), when \( u_a > u^* \), must be located in the \( z_1 \)-axis on which \( w = 0 \). In the Case 1, \( x_1^0(u_a) > 0 \) which implies that the \( z_1 \) coordinate of \( x^*(u_a) \) is strictly positive, which we denote by \( \alpha(u_a) > 0 \). From Assumption 3, it can also be shown (like in [16]) that \( h_{av,1}(z_1, 0, u_a) > 0 \) for \( z_1 \in (0, \alpha(u_a)) \) and \( h_{av,1}(z_1, 0, u_a) < 0 \) for \( z_1 > \alpha(u_a) \). This format suggests that the solution \( z_1(t) \) and \( w(t) \) may remain around the equilibrium \( (\alpha(u_a), 0) \) if its initial condition is around the equilibrium. We will make use of this property shortly.
From now on, we apply the coordinate change (25) to the full system (16) (through (21)) to obtain

\[
\begin{align*}
\dot{z}_1 &= h_{av,1}(z_1, w, u_a) + h_{p,1}(z_1, w, u_a, uf(t/\epsilon)) \\
\dot{w} &= A_2(u_a)w + N_2(z_1, w, u_a) + h_{p,2}(z_1, w, u_a, uf(t/\epsilon))
\end{align*}
\]

(28)

in which,

\[
\begin{align*}
h_{p,1}(z_1, w, u_a, uf) &= g_{p,1}(z_1, w + \pi(z_1, u_a), u_a, uf) \\
h_{p,2}(z_1, w, u_a, uf) &= g_{p,2}(z_1, w + \pi(z_1, u_a), u_a, uf) \quad \text{(29)}
\end{align*}
\]

where \( \int_0^T h_{p,i}(z_1, w, u_a, uf(s))ds = 0 \) (i = 1, 2). Here, we note that the line \( z_1 = 0 \) is the boundary of the positive orthant since \( z_1 = x_k \). In addition, we observe that

\[
h_{p,1}(z_1, w, u_a, uf) = z_1 \tilde{h}_{p,1}(z_1, w, u_a, uf).
\]

(30)

Now defining \( \tau = t/\epsilon \), we have that

\[
\begin{align*}
\frac{dz_1}{d\tau} &= \epsilon h_{av,1}(z_1, w, u_a) + \epsilon h_{p,1}(z_1, w, u_a, uf(\tau)) \\
\frac{dw}{d\tau} &= \epsilon(A_2(u_a)w + N_2(z_1, w, u_a)) + \epsilon h_{p,2}(z_1, w, u_a, uf(\tau)).
\end{align*}
\]

(31)

We apply the following coordinate change between \((z_1, w)\) and \((\tilde{z}_1, \tilde{w})\) (implicitly defined for \((\tilde{z}_1, \tilde{w})\))

\[
\begin{bmatrix}
\tilde{z}_1 \\
\tilde{w}
\end{bmatrix} = \begin{bmatrix}
z_1 \\
w
\end{bmatrix} + \epsilon \int_0^\tau \begin{bmatrix}
h_{p,1}(\tilde{z}_1, \tilde{w}, u_a, uf(s)) \\
h_{p,2}(\tilde{z}_1, \tilde{w}, u_a, uf(s))
\end{bmatrix} ds =: \begin{bmatrix}
z_1 \\
w
\end{bmatrix} + \epsilon \begin{bmatrix}
\mu_1(\tilde{z}_1, \tilde{w}, u_a, \tau) \\
\mu_2(\tilde{z}_1, \tilde{w}, u_a, \tau)
\end{bmatrix},
\]

(32)

which is borrowed from the averaging theory [4,15]. Note that \( \mu_1(\tilde{z}_1, \tilde{w}, u_a, \tau) = \tilde{z}_1 \tilde{\mu}_1(\tilde{z}_1, \tilde{w}, u_a, \tau) \) due to (30), and let \( \mu := [\mu_1, \mu_2]^T \). Taking derivatives of both sides with respect to \( \tau \), we obtain that

\[
\left( I + \epsilon \frac{\partial \mu(\tilde{z}_1, \tilde{w}, u_a, \tau)}{\partial (\tilde{z}_1, \tilde{w})} \right) \left[ \frac{d\tilde{z}_1}{d\tau} \frac{d\tilde{w}}{d\tau} \right] = \begin{bmatrix}
\epsilon h_{av,1}(\tilde{z}_1, \tilde{w}, u_a) + \epsilon h_{1}(\tilde{z}_1, \tilde{w}, u_a, uf, \epsilon) \\
\epsilon(A_2(u_a)\tilde{w} + N_2(\tilde{z}_1, \tilde{w}, u_a)) + \epsilon h_{2}(\tilde{z}_1, \tilde{w}, u_a, uf, \epsilon)
\end{bmatrix}
\]

(33)

where

\[
\tilde{h}_1 = h_{av,1}(\tilde{z}_1 + \epsilon \mu_1, \tilde{w} + \epsilon \mu_2, u_a) + h_{p,1}(\tilde{z}_1 + \epsilon \mu_1, \tilde{w} + \epsilon \mu_2, u_a, uf) \\
- h_{av,1}(\tilde{z}_1, \tilde{w}, u_a) - h_{p,1}(\tilde{z}_1, \tilde{w}, u_a, uf)
\]

\[
\tilde{h}_2 = A_2(u_a)\mu_2 + N_2(\tilde{z}_1 + \epsilon \mu_1, \tilde{w} + \epsilon \mu_2, u_a) + h_{p,2}(\tilde{z}_1 + \epsilon \mu_1, \tilde{w} + \epsilon \mu_2, u_a) \\
- N_2(\tilde{z}_1, \tilde{w}, u_a) - h_{p,2}(\tilde{z}_1, \tilde{w}, u_a, uf).
\]
Again, we note that \( \tilde{h}_1(\tilde{z}_1, \tilde{w}, u_a, u_f, \epsilon) = \tilde{z}_1 \tilde{h}_1(\tilde{z}_1, \tilde{w}, u_a, u_f, \epsilon) \). In addition, if we define a row vector function \( m = [m_1, m_2] \) as

\[
(I + \epsilon \frac{\partial \mu}{\partial (\tilde{z}_1, \tilde{w})})^{-1} = \begin{bmatrix}
1 + \epsilon \tilde{z}_1 \frac{\partial \mu_1}{\partial \tilde{z}_1} + \epsilon \tilde{z}_1 \frac{\partial \mu_1}{\partial \tilde{w}} & 0 \\
0 & 1 + m_1(\tilde{z}_1, \tilde{w}, u_a, \tau, \epsilon) & m_2(\tilde{z}_1, \tilde{w}, u_a, \tau, \epsilon)
\end{bmatrix}^{-1}
\]

then \( m_i \) has the property that \( m_i(\tilde{z}_1, \tilde{w}, u_a, \tau, 0) = 0 \) for \( i = 1, 2 \) and \( m_2(\tilde{z}_1, \tilde{w}, u_a, \tau, \epsilon) = \tilde{z}_1 m_2(\tilde{z}_1, \tilde{w}, u_a, \tau, \epsilon) \). (The above matrix becomes invertible with sufficiently small \( \epsilon \) in a local neighborhood of the origin.) This can be simply proved by putting \( \epsilon = 0 \) (so that the matrix becomes block-diagonal whose inverse should also be block-diagonal) and \( \tilde{z}_1 = 0 \), respectively.

Therefore, we obtain from (33) that

\[
\begin{align*}
\frac{d\tilde{z}_1}{d\tau} &= \epsilon [h_{av,1}(\tilde{z}_1, \tilde{w}, u_a) + H_1(\tilde{z}_1, \tilde{w}, u_a, u_f, \epsilon)] \\
\frac{d\tilde{w}}{d\tau} &= \epsilon [A_2(u_a)\tilde{w} + N_2(\tilde{z}_1, \tilde{w}, u_a) + H_2(\tilde{z}_1, \tilde{w}, u_a, u_f, \epsilon)]
\end{align*}
\]

(34)
in which, \( H_i(\tilde{z}_1, \tilde{w}, u_a, u_f, 0) = 0 \) for \( i = 1, 2 \), and \( H_1(\tilde{z}_1, \tilde{w}, u_a, u_f, \epsilon) = \tilde{z}_1 H_1(\tilde{z}_1, \tilde{w}, u_a, u_f, \epsilon) \).

We also note by (32) that the line \( \tilde{z}_1 = 0 \) corresponds to the boundary of the positive orthant in the \( x \)-coordinates.

Now suppose that we choose \( u^*_+ \) that is slightly larger than \( u^* \) and fix \( u_a = u^*_a \). Let \( P = P^T > 0 \) be the solution of \( A_2^T(u^*_+)P + PA_2(u^*_+) = -I \), and define a set

\[
\Omega(\bar{r}_1, \bar{r}_2) := \{(\tilde{z}_1, \tilde{w}) : 0 < \tilde{z}_1 < \bar{r}_1, \sqrt{\tilde{w}^T P \tilde{w}} < \bar{r}_2\}.
\]

It can be shown that this set is forward invariant for (34) with sufficiently small positive \( \bar{r}_1, \bar{r}_2, \) and \( \epsilon \), and with suitably chosen \( u^*_+ \) such that \( 0 < \alpha(u^*_+) < \bar{r}_1 \). Indeed, the boundary of \( \Omega \) with \( \tilde{z}_1 = 0 \) cannot be crossed because \( d\tilde{z}_1/d\tau \) is zero there. Moreover, at the boundary with \( \tilde{z}_1 = \bar{r}_1, \) \( d\tilde{z}_1/d\tau \) is negative with sufficiently small \( \bar{r}_2 \) because of the fact that \( h_{av,1}(\tilde{z}_1, 0, u^*_+) < 0 \) if \( \tilde{z}_1 > \alpha(u^*_+) \). Finally, at the boundary where \( \sqrt{\tilde{w}^T P \tilde{w}} = \bar{r}_2 \), we have that

\[
\frac{1}{\epsilon} \frac{d\sqrt{\tilde{w}^T P \tilde{w}}}{d\tau} \leq - \frac{1}{2\sqrt{\lambda_{\text{max}}(P)}} \| \tilde{w} \| + \frac{k_2(\tilde{z}_1, \tilde{w})\lambda_{\text{max}}(P)}{\sqrt{\lambda_{\text{min}}(P)}} \| \tilde{w} \| + \frac{\lambda_{\text{max}}(P)}{\sqrt{\lambda_{\text{min}}(P)}} \| H_2(\tilde{z}_1, \tilde{w}, u^*_+, u_f, \epsilon) \| \tag{35}
\]

where \( k_2 \) is a continuous function such that \( k_2(0, 0) = 0 \) and \( \| N_2(\tilde{z}_1, \tilde{w}, u^*_+) \| \leq k_2(\tilde{z}_1, \tilde{w}) \| \tilde{w} \| \). Since \( \| H_2 \| \) can be made arbitrarily small with sufficiently small \( \epsilon \) in the set \( \Omega(\bar{r}_1, \bar{r}_2) \), it follows that the right-hand side of the above
inequality becomes negative at the boundary where $\sqrt{\mathbf{w}^T P \mathbf{w}} = r_2$ with sufficiently small $\bar{r}_1$ and $\bar{r}_2$. In addition, it is seen that, as $\epsilon \to 0$, the solution $\mathbf{w}(t)$ converges to an arbitrarily small neighborhood of the $\tilde{z}_1$-axis.

Finally, let $V = (1/2)(\tilde{z}_1 - \alpha(u^*_+))^2 + \sqrt{\mathbf{w}^T P \mathbf{w}}$. Then,

$$
\frac{1}{\epsilon} \frac{dV}{dt} \leq (\tilde{z}_1 - \alpha(u^*_+))h_{av,1}(\tilde{z}_1, 0, u^*_+) + (\tilde{z}_1 - \alpha(u^*_+))N_1(\tilde{z}_1, \mathbf{w}, u^*_+)
- \frac{1}{2\sqrt{\lambda_{\max}(P)}} \|\mathbf{w}\| + \frac{k_2(\tilde{z}_1, \mathbf{w})}\lambda_{\max}(P) \|\mathbf{w}\|
+ \left[(\tilde{z}_1 - \alpha(u^*_+))H_1(\tilde{z}_1, \mathbf{w}, u^*_+, u_f, \epsilon) + \frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}} \|H_2(\tilde{z}_1, \mathbf{w}, u^*_+, u_f, \epsilon)\|\right]
$$

where $N_1(\tilde{z}_1, \mathbf{w}, u^*_+) = h_{av,1}(\tilde{z}_1, \mathbf{w}, u^*_+) - h_{av,1}(\tilde{z}_1, 0, u^*_+)$, and thus, satisfies that $\|N_1(\tilde{z}_1, \mathbf{w}, u^*_+)\| \leq k_1(\tilde{z}_1, \mathbf{w})\|\mathbf{w}\|$ with a continuous function $k_1$ such that $k_1(0, 0) = 0$. Therefore, there are class-$\mathcal{K}$ functions $\gamma_1$ and $\gamma_2$ such that

$$
\frac{1}{\epsilon} \frac{dV}{dt} \leq -\gamma_1 (\|\tilde{z}_1 - \alpha(u^*_+)\|) + \gamma_2(\epsilon) \quad \text{on the set } (\Omega(\tilde{r}_1, \tilde{r}_2) - \Omega(r_1, r_2))
$$

with some $r_1$ and $r_2$ such that $0 < r_1 < \alpha(u^*_+) < \tilde{r}_1$ and $0 < r_2 < \tilde{r}_2$. See Fig. 3.

**Remark 2.** The reason why the inequality (36) does not hold in the set $\Omega(r_1, r_2)$ is that the term $\tilde{z}_1 - \alpha(u^*_+)h_{av,1}(\tilde{z}_1, 0, u^*_+)$ approaches to zero as $\tilde{z}_1$ tends to zero, and thus, we deducted $\Omega(r_1, r_2)$ from $\Omega(\tilde{r}_1, \tilde{r}_2)$. Therefore, if any solution in the set $\Omega(r_1, r_2)$ escapes from this set through its right boundary, then the solution initiated in $\Omega(\tilde{r}_1, \tilde{r}_2)$ will tend to a small neighborhood of $(\alpha(u^*_+), 0)$ whose size vanishes as $\epsilon \to 0$.

Since any solution in $\Omega(r_1, r_2)$ cannot cross its boundary upwards or downwards due to (35), we now prove that the solution crosses the right
boundary. For this, we observe from (34) and (27) that
\[
\frac{d\tilde{z}}{dt} = \epsilon[H_1(\tilde{z}, \tilde{w}, u^*_1, u_f, \epsilon)]
\]
This results in (37).

Recall the fact that \(R_1(0, 0, u^*_1) = 0\) and \(\tilde{H}_1(\tilde{z}, \tilde{w}, u^*_1, u_f, 0) = 0\). Since \(A_1(u^*_1) > 0\), with sufficiently small positive \(r_1, r_2\), and \(\epsilon\), the state \(\tilde{z}(t)\) keeps increasing and thus the solution will escape from \(\Omega(r_1, r_2)\) through its right boundary.

Now we return to the Jump & Wait strategy. Let \(\sigma_+ = u^*_m - u^*_s\). And, by suitably choosing \(\sigma_-\) and \(\rho_-\), the set \(B_{\rho_-}(x^*(u^* - \sigma_-)) \cap \mathbb{R}^n_+\) in the x-coordinates is located inside the set \(\Omega(\tilde{r}_1, \tilde{r}_2)\) in the \((\tilde{z}, \tilde{w})\)-coordinates.

Then, it is seen from (36) that the state \(x(t^*) \in B_{\rho_-}(x^*(u^* - \sigma_-)) \cap \mathbb{R}^n_+\) can be driven to \(B_{\rho_+}(x^*(u^* + \sigma_+)) \cap \mathbb{R}^n_+\) by keeping \(u_a = u^* + \sigma_+\) and waiting with sufficiently small \(\epsilon\).

On the other hand, in the Case 2, the difficulty in Remark 2 does not arise because the equilibrium \(x^*(u_a)\) leaves the closure of the positive orthant (since \(x^*_1(u_a) < 0\)) when \(u_a > u^*\). Therefore, the weak negativity caused by the existence of the equilibrium at the boundary is not the case, and the Jump & Wait strategy still does apply.

**Example.** Consider again the system (12) with \(u = u_a + u_f\). Let the fast input \(u_f(\tau)\) is given by 0.1 \(\sin(\tau)\). Then, it can be rewritten as
\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{bmatrix}
u_a - x - u_a xy \\
u_a xy - y
\end{bmatrix} + \begin{bmatrix}u_f - u_f xy \\
u_f xy
\end{bmatrix} = f_{av}(x, y, u_a) + f_p(x, y, u_a, u_f).
\]

This system and the curve of equilibria \(x^*\) in (13) satisfy all the assumptions in this section with \(k = 2\).

To verify the result, a computer simulation is performed with the parameters \(\bar{a}_1 = 0.9, \bar{a}_2 = 1.1, \sigma_- = \sigma_+ = 0.05, \kappa_- = \kappa_+ = 0.0005, \epsilon = 20, \) and \(T_{\text{Wait}} = 300\). With the initial condition \(x = 1\) and \(y = 0.1\), and the input \(u_a(t)\) is linear with the slope \(\kappa_- = \kappa_+\) or 0, the result is depicted in Fig. 4.

## 5 Conclusion

In practice, there are several occasions where the slowly-varying-average-input assumption is more suitable than the slowly-varying-input assumption. A classical example of the slowly-varying-average-input is the PWM (pulse-width modulation) control whose net effect is the average of instantaneous control inputs. Another example is the drug dose control for a patient. In fact, the drug effect in the body reaches its maximum immediately after a patient takes the medicine and then is going down until the patient takes another next time. However, since it is too complicated to consider the
instantaneous variation of the drug effect, a long-term planning of drug dose schedule does not usually take it into account, assuming that the net effect of the drug will be the average of instantaneous drug effect.

In this paper we have presented three extended results on the stability property with slowly varying inputs that has been discussed in, e.g., Khalil and Kokotović [9]. Our first extension shows that the inputs may vary fastly, but once its average is slowly varying, then the same stability property still holds. The second extension deals with the case when a bifurcation occurs so that the assumptions of the stability property are violated. We have proposed a Jump & Wait strategy which guarantees the stability property even under this case. The final extension is to combine both extensions; that is, the Jump & Wait strategy is employed not with the input itself but with the average of the input.

References


**A Proof of Theorem 2**

The following proof is taken from [3] for readers’ convenience by making the paper self-contained.
First of all, let us define a compact set of our interest,
\[ D := \{ (x, u_a) \in \mathbb{R}^n \times \Gamma : x \in \bar{B}_R(x^*(u_a)) \}. \]

For each element \((x, u_a) \in D\), define \(\mu(x, u_a, \tau)\) as
\[ \mu(x, u_a, \tau) = \int_0^\tau f_p(x, u_a, u_f(s)) ds \]
where \(\tau \in \mathbb{R}\). Thus, obviously, it follows that
\[ \frac{\partial \mu}{\partial x}(x, u_a, \tau) = \int_0^\tau \frac{\partial f_p}{\partial x}(x, u_a, u_f(s)) ds, \]
and
\[ \frac{\partial \mu}{\partial u_a}(x, u_a, \tau) = \int_0^\tau \frac{\partial f_p}{\partial u_a}(x, u_a, u_f(s)) ds. \]

Note that the above integrals are conducted with \(x\) and \(u_a\) fixed. The norms of \(\mu(x, u_a, \tau), \frac{\partial \mu}{\partial x}(x, u_a, \tau), \) and \(\frac{\partial \mu}{\partial u_a}(x, u_a, \tau)\) are bounded by some constant \(c_m > 0\) for all \((x, u_a, \tau) \in D \times [0, \infty)\) since \(f_p(x, u_a, u_f(\tau))\) and its partial derivatives with respect to \(x\) and \(u_a\) are \(T\)-periodic in \(\tau\) and have zero mean.

Consider the change of variables
\[ x = y + \epsilon \mu(y, u_a, t/\epsilon), \]
where the new variable \(y\) is an \(n\)-dimensional vector. Note that the Jacobian of the map \((38)\) is \(\frac{\partial x}{\partial y}(y, u_a, \tau) = I + \epsilon \frac{\partial \mu}{\partial y}(y, u_a, \tau)\). Even though \(y\) is defined implicitly, the Jacobian is nonsingular, and the ratio of each leading principal minor of the Jacobian is strictly positive on \(D \times [0, \infty)\) for sufficiently small \(\epsilon\) since the partial derivative \(\frac{\partial \mu}{\partial y}\) is bounded on \(D \times [0, \infty)\) as stated earlier. Then, owing to [11, Thm. 1], there exists an \(\epsilon_1 > 0\) such that, for each \((u_a, t, \epsilon) \in \Gamma \times [0, \infty) \times [0, \epsilon_1]\), the map \(y \mapsto x\) is \(C^1\) and bijective for all \(y \in \bar{B}_R(x^*(u_a))\). Differentiating both sides of the equation \((38)\) with respect to time, we have
\[
\dot{x}(t) = \dot{y}(t) + \epsilon \frac{\partial \mu}{\partial y}(y(t), u_a(t), t/\epsilon) \cdot \dot{y}(t) + \epsilon \frac{\partial \mu}{\partial u_a}(y(t), u_a(t), t/\epsilon) \cdot \dot{u}_a(t) \\
+ \epsilon \frac{\partial \mu}{\partial \tau}(y(t), u_a(t), t/\epsilon) \cdot \frac{1}{\epsilon}.
\]
The above equation is rewritten as

\[
\begin{align*}
\left[ I + \epsilon \frac{\partial \mu}{\partial y} (y(t), u_a(t), t/\epsilon) \right] \hat{y}(t) &= \dot{x}(t) - \epsilon \frac{\partial \mu}{\partial u_a}(y(t), u_a(t), t/\epsilon) \cdot \hat{u}_a(t) - \frac{\partial \mu}{\partial \tau}(y(t), u_a(t), t/\epsilon) \\
&= f(x(t), u_a(t), uf(t/\epsilon)) - \epsilon \frac{\partial \mu}{\partial u_a}(y(t), u_a(t), t/\epsilon) \cdot \hat{u}_a(t) \\
&\quad - f_p(y(t), u_a(t), uf(t/\epsilon)) \\
&= f_{av}(y(t), u_a(t)) - \epsilon \frac{\partial \mu}{\partial u_a}(y(t), u_a(t), t/\epsilon) \cdot \hat{u}_a(t) \\
&\quad + f(x(t), u_a(t), uf(t/\epsilon)) - f(y(t), u_a(t), uf(t/\epsilon)) \\
&= f_{av}(y, u_a) + p_0(y, u_a, t, \epsilon) \hat{u}_a + p_1(y, u_a, uf, t, \epsilon),
\end{align*}
\]

where \( p_0(y, u_a, t, \epsilon) = -\epsilon \frac{\partial \mu}{\partial u_a} \) and \( p_1(y, u_a, uf, t, \epsilon) = [f(y + \epsilon \mu, u_a, uf) - f(y, u_a, uf)] \). The function \( p_1 \) is written as

\[
p_1(y, u_a, uf, t, \epsilon) = F(y, u_a, uf, \epsilon \mu) \cdot \epsilon \mu,
\]

where \( F \) is a matrix of continuous functions whose existence follows from the continuous differentiability of \( f \). It can be seen easily that

\[
\|p_0(y, u_a, t, \epsilon)\| \leq \epsilon c_m \tag{39} \\
\|p_1(y, u_a, uf, t, \epsilon)\| \leq \epsilon_c c_m \tag{40}
\]

for all \((y, u_a) \in D, t \in [0, \infty)\) and \( \epsilon \in [0, \epsilon_1]\), where

\[
c_F = \sup_{(y, u_a) \in D, 0 \leq \tau \leq t \leq \epsilon \mu \leq \epsilon_1 c_m} \|F(y, u_a, uf(\tau), \epsilon \mu)\|
\]

\((c_F < \infty \text{ since } F \text{ is continuous and } uf \text{ is uniformly bounded}).

The function \( V \) in Assumption 1 is used to analyze the transformed system. The time-derivative of \( V \) along the system in \( y \)-coordinates is given by

\[
\dot{V}(y, u_a) = \frac{\partial V}{\partial y} \hat{y}(t) + \frac{\partial V}{\partial u_a} \hat{u}_a(t)
\]

\[
= \frac{\partial V}{\partial y} \left[ I + \epsilon \frac{\partial \mu}{\partial y} \right]^{-1} (f_{av} + p_0 \hat{u}_a + p_1) + \frac{\partial V}{\partial u_a} \hat{u}_a(t)
\]

\[
= \frac{\partial V}{\partial y} f_{av} + \frac{\partial V}{\partial y} \left( \left[ I + \epsilon \frac{\partial \mu}{\partial y} \right]^{-1} - I \right) f_{av}
\]

\[
+ \frac{\partial V}{\partial y} \left[ I + \epsilon \frac{\partial \mu}{\partial y} \right]^{-1} (p_0 \hat{u}_a + p_1) + \frac{\partial V}{\partial u_a} \hat{u}_a(t).
\]
Since the function $V$ is locally Lipschitz, its partial derivatives $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial u_a}$ are bounded on $D$ almost everywhere, and we let

$$
c_v = \text{ess.sup}_{(y,u_a) \in D} \left\| \frac{\partial V}{\partial y} (y,u_a) \right\|,
$$

$$
c_w = \text{ess.sup}_{(y,u_a) \in D} \left\| \frac{\partial V}{\partial u_a} (y,u_a) \right\|,
$$

$$
c_a = \text{ess.sup}_{(y,u_a) \in D} \left\| f_{av}(y,u_a) \right\|.
$$

In addition, for $\epsilon < \frac{1}{c_m}$, we have

$$
\left\| \left[ I + \epsilon \frac{\partial \mu}{\partial y} \right]^{-1} \right\| \leq \frac{1}{1 - \epsilon c_m} \quad \text{and} \quad \left\| \left[ I + \epsilon \frac{\partial \mu}{\partial y} \right]^{-1} - I \right\| \leq \frac{\epsilon c_m}{1 - \epsilon c_m}.
$$

Consequently, from Assumption 1, (39) and (40), it follows that

$$
\dot{V}(y(t), u_a(t)) \leq -\alpha_3(\|y(t) - x^*(u_a)\|) + c_v \frac{\epsilon c_m}{1 - \epsilon c_m} c_a + c_v \frac{1}{1 - \epsilon c_m} \epsilon c_m \|\dot{u}_a\| + c_w \frac{1}{1 - \epsilon c_m} \epsilon c_F c_m + c_w \|\dot{u}_a\|,
$$

on $D$, almost everywhere.

Now, without loss of generality, suppose that the given number $\rho \leq R$. We can take a pair of positive numbers $(\epsilon_2, \kappa)$ such that

$$
\frac{\epsilon_2 c_m}{1 - \epsilon_2 c_m} c_v(c_a + \kappa + \epsilon_F) + c_w \kappa \leq (\alpha_3 \circ \alpha_2^{-1} \circ \alpha_1)(\frac{\rho}{2}).
$$

Then,

$$
\dot{V}(y(t), u_a(t)) \leq -\alpha_3(\alpha_2^{-1}(V)) + (\alpha_3 \circ \alpha_2^{-1} \circ \alpha_1)(\frac{\rho}{2}),
$$

which implies that, if $V(y(0), u_a(0)) \leq \alpha_1(\rho/2)$, then $V(y(t), u_a(t)) \leq \alpha_1(\rho/2)$ for all $t \geq 0$. Therefore, letting $\delta = (\alpha_2^{-1} \circ \alpha_1)(\rho/2)$, we have

$$
\|y(t) - x^*(u_a(t))\| \leq \frac{\rho}{2},
$$

for all $t \geq 0$ if $\|y(0) - x^*(u_a(0))\| \leq \delta$ and $\epsilon < \min\{\epsilon_1, \epsilon_2, 1/c_m\}$.

To complete the proof, the analysis is given in $x$-coordinate. From (38), $x(0) = y(0)$. Thus, if $\|x(0) - x^*(u_a(0))\| \leq \delta$, then $\|y(t) - x^*(u_a(t))\| \leq \delta/2$ and $(y(t), u_a(t)) \in D$ for all $t \geq 0$, which implies that

$$
\|x(t) - x^*(u_a(t))\| \leq \|x(t) - y(t)\| + \|y(t) - x^*(u_a)\| \leq \|\epsilon \mu(y(t), u_a(t), t/\epsilon)\| + \frac{\rho}{2} \leq \epsilon c_m + \frac{\rho}{2}.
$$

Therefore, with $\epsilon^* := \min\{\epsilon_1, \epsilon_2, 1/c_m, \rho/(2c_m)\}$, we have that $\|x(t) - x^*(u_a(t))\| \leq \rho$ for all $t \geq 0$ provided that $\epsilon < \epsilon^*$.
B  Existence of the Coordinate Change for (20)

With

\[ f_{av,k}(x, u_a) = x_k \tilde{f}_{av,k}(x, u_a), \]

the \( k \)-th row of the Jacobian is

\[
\frac{\partial}{\partial x} f_{av,k}(x, u_a) = \left[ x_k \frac{\partial f_{av,k}}{\partial x_1}, \ldots, \tilde{f}_{av,k}(x, u_a) + x_k \frac{\partial f_{av,k}}{\partial x_k}, \ldots, x_k \frac{\partial f_{av,k}}{\partial x_n} \right].
\]

Evaluating it at \( x^o(u_a) \) (with the property that \( x^o_k(u_a) = 0 \) locally around \( u^* \)), we obtain that

\[
\frac{\partial}{\partial x} f_{av,k}(x^o(u_a), u_a) = [0, \ldots, 0, \tilde{f}_{av,k}(x^o(u_a), u_a), 0, \ldots, 0]
\]

that is, only the \( k \)-th element is \( \tilde{f}_{av,k}(x^o(u_a), u_a) = A_1(u_a) \) while the others are zero. Now, we show the existence of a coordinate change

\[
\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = T(u_a)(x - x^o(u_a))
\]

such that \( z_1 = x_k \) regardless of \( u_a \), which converts \( \dot{x} = f_{av}(x, u_a) \) into \( \dot{z} = g_{av}(z, u_a) \) whose Jacobian is

\[
\begin{bmatrix} A_1(u_a) & 0 \\ 0 & A_2(u_a) \end{bmatrix}
\]

with the scalar \( A_1(u^*) = 0 \) and the \( (n - 1) \times (n - 1) \) matrix \( A_2(u_a) \) is nonsingular locally around \( u^* \). For this, we first take any invertible \( n \times n \) matrix \( T_1(u_a) \) such that its first row is \( e^T_1 \) (so that \( z = T_1(u_a)(x - x^o(u_a)) \) gives that \( z_1 = x_k \)). Then, the \( k \)-th row of \( T_1^{-1}(u_a) \) should be \( e^T_k \) (because \( x_k = z_1 \) by the coordinate change using \( T_1(u_a) \)). Therefore, it can be seen by computation that

\[
T_1(u_a) \frac{\partial f_{av}}{\partial x}(x^o(u_a), u_a) T_1^{-1}(u_a) = \begin{bmatrix} A_1(u_a) & 0_{1 \times (n-1)} \\ A_{21}(u_a) & A_2(u_a) \end{bmatrix}
\]

in which, \( A_1 \) has already been defined, and \( A_{21} \) and \( A_2 \) are suitably defined here. Note that \( A_1(u^*) = 0 \) (from the assumption) and \( A_2(u_a) \) is nonsingular locally around \( u^* \) by the invertibility of \( T_1 \). Now solve \( S(u_a) \) in the Sylvester equation

\[
S(u_a)A_1(u_a) - A_2(u_a)S(u_a) = -A_{21}(u_a).
\]

Since \( A_1(u_a) \) and \( -A_2(u_a) \) have distinct eigenvalues around \( u^* \), the solution \( S(u_a) \) exists around \( u^* \). Now, let

\[
T_2(u_a) = \begin{bmatrix} 1 & 0_{1 \times (n-1)} \\ S(u_a) & I \end{bmatrix} T_1(u_a).
\]
Then, since it is verified that
\[
\begin{bmatrix}
1 & 0 \\
S(u_a) & I
\end{bmatrix}
\begin{bmatrix}
A_1(u_a) & 0 \\
A_2(u_a)
\end{bmatrix}
= 
\begin{bmatrix}
A_1(u_a) & 0 \\
0 & A_2(u_a)
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
S(u_a) & I
\end{bmatrix},
\]
the matrix \(T(u_a) := T_2(u_a)\) does the job.

C Derivation for (23)

The system (22) is rewritten as
\[
\begin{align*}
\dot{u}_a &= 0 \\
\dot{z}_1 &= g_{av,1}(z, u_a) =: G_{av,1}(z, u_a) \\
\dot{z}_2 &= A_2(u^*)z_2 + [g_{av,2}(z, u_a) - A_2(u^*)z_2] =: A_2(u^*)z_2 + G_{av,2}(z, u_a).
\end{align*}
\]
Here, we have that
\[
G_{av,i}(0, u_a) = 0, \quad \text{for all } u_a \in [u^*, u_2^*], \quad \frac{\partial G_{av,i}}{\partial z}(0, u^*) = 0
\]
for \(i = 1, 2\). Since this is the standard format for the center manifold theory [10], it can be seen that there exists a function \(\pi\) in a neighborhood of the origin such that
\[
\pi(0, u^*) = 0, \quad \frac{\partial \pi}{\partial z_1}(0, u^*) = 0
\]
\[
A_2(u^*)\pi(z_1, u_a) + G_{av,2}(z_1, \pi(z_1, u_a), u_a) = \frac{\partial \pi}{\partial z_1} G_{av,1}(z_1, \pi(z_1, u_a), u_a).
\]
In fact, by [16, Claim 2], we even have the property \(\pi(0, u_a) = 0\) when \(u_a\) is locally around \(u^*\). In [16], the set \(\{(z_1, z_2) : z_2 = \pi(z_1, u_a)\}\) is called ‘parametrized center manifold.’

Received September 2009.

http://monotone.uwaterloo.ca/~journal/