Practical Consensus for Heterogeneous Linear Time-Varying Multi-Agent Systems

Jaeyong Kim\(^1\), Jongwook Yang\(^1\), Jungsu Kim\(^2\), and Hyungbo Shim\(^1\)*

\(^1\)ASRI, Department of Electrical Engineering, Seoul National University, Seoul 151-741, Korea
(Tel : +82-2-880-1786; E-mail: jykim@cdsl.kr, \{wookchan, hshim\}@snu.ac.kr) * Corresponding author
\(^2\)Department of Electrical Engineering and Information Technology, Seoul National University of Science and Technology, Seoul 139-743, Korea
(Tel : +82-2-970-6547; E-mail: jungsu@snu.ac.kr)

Abstract: This paper studies practical consensus (PC) problem for linear multi-agent systems (MAS) under a fixed, connected, and undirected communication network. PC means that the difference between any two agents remains in an arbitrarily chosen neighborhood of the origin after some time by using a suitable control law. All the agents, which are heterogeneous first-order linear time-varying (LTV) systems, can collect only relative state information from their neighborhoods. According to the assumptions of MAS, we divide the problem into two cases; one is a class of homogeneous LTV MAS with heterogeneous disturbances where the difference between any two disturbances is bounded, and the other is a class of heterogeneous LTV MAS with heterogeneous bounded disturbances. Finally, we show that PC for each case is always achieved if the control gain is sufficiently large.

Keywords: practical consensus, heterogeneous multi-agent systems, linear time-varying systems.

1. INTRODUCTION

Consensus and synchronization appear in various areas of biology, social sciences, engineering, etc. Flocking of birds, schooling of fish, and swarming of ants and honeybees are typical in nature [1, 2]. Sometimes, the consensus theory is a useful tool for understanding social phenomena [3]. In recent years, lots of studies have been made on consensus problems of multi-agent systems (MAS) in the field of engineering [4–8]. Most of them have considered the homogeneous linear time-invariant (LTI) cases, i.e., the cases where all agents are identical LTI systems. In reality, however, the dynamics of all agents may not only be different from each other, but also depend on time.

Fortunately, there have been several remarkable studies regarding the heterogeneous LTI cases [9–11]. Zhao et al. [9] solved the problem of synchronization for heterogeneous LTI nonlinear MAS by introducing the average dynamics of all agents. Wieland et al. [10] showed that an internal model principle is a necessary and sufficient condition for output feedback consensus of heterogeneous LTI MAS. Kim et al. [11] solved the output consensus problem of heterogeneous and uncertain LTI MAS by designing the internal models. Although they all contributed to solve the consensus problem of heterogeneous MAS, the results have still focused on the LTI MAS. For linear time-varying (LTV) MAS, some sufficient conditions have been presented, but limited to the homogeneous cases [12].

In this respect, we study the consensus problem of heterogeneous LTV MAS. As the first attempt toward this problem, we deal with two cases where each agent, for which the static diffusive coupling [4–6] is employed, is a LTV scalar system perturbed by disturbance. In the first case, all agents are identical except for disturbances where the difference between any two disturbances is bounded, and while in the second case, non-identical agents with bounded disturbances are considered.

By the internal model principle [10], there must exist at least one common mode in the closed loop system of each agent together with the dynamic coupling using even self output feedback even for unperturbed heterogeneous LTI MAS to achieve asymptotic consensus (AC), i.e., the usual concept of consensus so far; in particular, the dynamic coupling and the self output feedback play a crucial role in giving rise to the common mode. Moreover, since this paper addresses perturbed heterogeneous LTV scalar systems, we can deduce that it is not easy to achieve AC of the considered systems using the static couplings without any self feedbacks. For this reason, we introduce the concept of practical consensus (PC), a practical version of AC, rather than consider the AC problem of heterogeneous LTV MAS.

Using the perturbation theory [13], we show that, for each case, the difference between any agents remains in an arbitrarily chosen neighborhood of the origin after some time if the coupling gain is sufficiently large. We also present the lower bounds for the coupling gains.

The remainder of this paper is organized as follows. Section II presents some basic definitions, network properties, and a suitable coordinate transformation for achieving PC. In Section III and IV, we derive sufficient conditions to achieve PC of homogeneous and heterogeneous perturbed LTV MAS, respectively. Finally, Section VI concludes the paper.

\[ \mathcal{G} = \]
\( (N, \mathcal{E}, \mathcal{A}) \), where \( N = \{1, 2, \ldots, N\} \) is a finite nonempty set of nodes, \( \mathcal{E} \subseteq N \times N \) an edge set of ordered pairs of nodes, and \( \mathcal{A} = \{a_{ij}\} \in \mathbb{R}^{N \times N} \) an adjacency matrix. An edge \((i, j) \in \mathcal{E}\), being represented by an arrow oriented toward \( j \) and tailed at \( i \), implies that the information flows from the node \( i \) to the node \( j \). A graph is said to be undirected if it has the property that \((i, j) \in \mathcal{E}\) implies \((j, i) \in \mathcal{E}\) for any node \( i, j \in N \). We exclude the self-connection, i.e., \((i, i) \notin \mathcal{E}\). It is related with \( \alpha_{ji} \) by the rule that \( \alpha_{ji} = 1 \) if and only if \((i, j) \in \mathcal{E}\). Otherwise \( \alpha_{ji} = 0 \). A path (of length \( l \)) from node \( i \) to node \( j \) is a sequence \( i_0, i_1, \ldots, i_l \) of nodes such that \( i_0 = i, i_l = j \), \((i_k, i_{k+1}) \in \mathcal{E}\), and \( i_k \)'s are distinct. An undirected graph \( \mathcal{G} \) is connected if there is a path from \( i \) to \( j \) for arbitrary two distinct nodes \( i, j \in N \).

The Laplacian matrix \( L = [l_{ij}] \in \mathbb{R}^{N \times N} \) of \( \mathcal{G} \) is defined as \( L := D - \mathcal{A} \), where the \( i \)th diagonal of the diagonal matrix \( D \) is given by \( d_i := \sum_{j \in N} \alpha_{ij} \). By its construction, it contains a zero eigenvalue with a corresponding eigenvector \( 1_N \) (i.e., \( L1_N = 0 \)) and all the other eigenvalues lie in the open right-half complex plane, where \( 1_N \) denotes the \( N \times 1 \) column vector comprising all ones. Thus, we sort them as \( 0 \leq \lambda_1 \leq \text{Re}(\lambda_2(L)) \leq \cdots \leq \text{Re}(\lambda_{N}(L)) \), where \( \lambda_i \)'s are eigenvalues of \( L \). The zero eigenvalue is simple if and only if the corresponding graph \( \mathcal{G} \) is connected [6]. For undirected graphs, both \( \mathcal{A} \) and \( L \) are symmetric so that \( \lambda_i \)'s are real numbers.

For matrices \( A_1, \ldots, A_k \), \( \text{diag}(A_1, \ldots, A_k) \) is defined as the block diagonal matrix whose \( i \)th diagonal entry is \( A_i \). For a vector \( x \) and a matrix \( A \), \( \|x\| \) and \( \|A\| \) denote the Euclidean norm and the induced matrix 2-norm, respectively.

### 2. PROBLEM STATEMENT

Consider a group of \( N \) heterogeneous first-order LTV MAS given as
\[
\dot{x}_i = a_i(t)x_i + b_i(t) + u_i, \quad x_i \in \mathbb{R}, \quad u_i \in \mathbb{R},
\]
where \( i \in N = \{1, 2, \ldots, N\} \) represents the agent’s identifier, \( a_i(t) \) and \( b_i(t) \) are continuous on \([t_0, \infty)\), and \( t_0 \) is the initial time. Furthermore, \( a_i(t) \) is bounded on \([t_0, \infty)\). Therefore, there exists \( M_i > 0 \) such that \( |a_i(t)| \leq M_i \) for all \( t \geq t_0 \) and \( i \in N \). The dynamics of each agent can be considered as LTV system \( \dot{x}_i = a_i(t)x_i \) with time-varying disturbance \( b_i(t) \).

The network topology is an undirected graph which is assumed to be fixed and connected.

Since the network topology is undirected graph, the Laplacian matrix \( L \) is symmetric [6]. Thus, by Schur’s lemma, there exists an orthogonal matrix \( \frac{1}{\sqrt{N}}U \) such that all columns of \( U \) are orthogonal and \( L = \frac{1}{\sqrt{N}}U \text{diag}(0, \Lambda)(\frac{1}{\sqrt{N}}U)^{-1} = U \text{diag}(0, \Lambda)U^{-1} \), where \( \Lambda \in \mathbb{R}^{(N-1) \times (N-1)} \) is a diagonal matrix having positive eigenvalue [14]. Without loss of generality, we can let the orthogonal matrix \( U \) be of the form
\[
U = \begin{bmatrix}
1 & Q_{12} \\
1_{N-1} & Q_{22}
\end{bmatrix}, \quad U^{-1} = \begin{bmatrix}
r_{11} & r_{12} \\
r_{21} & r_{22}
\end{bmatrix},
\]
where \( Q_{12} = \{q_{ij}\}_{i=2,\ldots,N} \in \mathbb{R}^{1 \times (N-1)}, \quad Q_{22} = \{q_{ij}\}_{i,j=2,\ldots,N} \in \mathbb{R}^{(N-1) \times (N-1)}, \quad r_{12} = \{r_{ij}\}_{i=2,\ldots,N} \in \mathbb{R}^{1 \times (N-1)}, \quad R_{21} = \{r_{ij}\}_{i=2,\ldots,N} \in \mathbb{R}^{(N-1) \times 1}, \quad R_{22} = \{r_{ij}\}_{i,j=2,\ldots,N} \in \mathbb{R}^{(N-1) \times (N-1)} \). Then, from \( U^{-1}U = I_N \) and the fact that all columns of \( \frac{1}{\sqrt{N}}U \) are orthonormal, we have these network properties such as
\[
\begin{align*}
R_{21} + R_{22}1_{N-1}^T & = 0_{(N-1) \times 1}, \\
R_{11}Q_{12} + R_{12}Q_{22} & = 0_{1 \times (N-1)}, \\
R_{12}Q_{21} + R_{22}Q_{22} & = I_{N-1}, \\
Q_{12} + 1^{T}_{N-1}Q_{22} & = 0_{1 \times (N-1)}.
\end{align*}
\]
From the network properties (2) and (5),
\[
\begin{align*}
\|R_{21}\| & = \|R_{22}1_{N-1}\| \leq \sqrt{N-1}\|R_{22}\|, \\
\|Q_{12}\| & = \|1^{T}_{N-1}Q_{22}\| \leq \sqrt{N-1}\|Q_{22}\|.
\end{align*}
\]
Also, from \( \left( \frac{1}{\sqrt{N}}U \right)^T = \left( \frac{1}{\sqrt{N}}U \right)^{-1} \),
\[
r_{1j} = \frac{1}{\sqrt{N}}, \quad \forall j \in N.
\]
Since \( r_{1j} < \frac{1}{\sqrt{N}} \leq 1 \) for all \( j \in N \) and (3),
\[
\begin{align*}
\|r_{11}Q_{12}\| & = \|R_{12}Q_{22}\| \leq \|R_{12}\|\|Q_{22}\| \\
& \leq \sqrt{N-1}\|Q_{22}\| \leq \sqrt{N-1}\|Q_{22}\|.
\end{align*}
\]
We assume that the agent \( i \) collects only the relative state information between the \( i \)th agent and its neighborhood by diffusive coupling [4–6],
\[
u_i = k \sum_{j=1}^{N} \alpha_{ij}(x_j - x_i), \quad (10)
\]
where \( k \in \mathbb{R} \) is a diffusive coupling gain which is applied to all agents.

**Definition 1:** The system (1) is said to achieve asymptotic consensus (AC) if there exists a diffusive coupling input (10) such that \( \lim_{t \to \infty} |x_i(t) - x_j(t)| = 0 \), \( \forall i, j \in N \).

**Definition 2:** The system (1) is said to achieve practical consensus (PC) if for any given \( \epsilon > 0 \), there exist a diffusive coupling input (10), and a real number \( T \geq 0 \) (dependent on \( \epsilon \) and \( x_i(t_0) \) for all \( i \in N \) such that \( |x_i(t) - x_j(t)| \leq \epsilon \), \( \forall t \geq t_0 + T, \forall i, j \in N \).

**Remark 1:** The notion of ‘practical consensus’ in Definition 2 differs from that of [15]. In Definition 2, the error \( |x_i(t) - x_j(t)| \) should be made arbitrarily small by choosing suitable \( u_i \)'s, however, the only interest in [15] is its boundedness. The dynamics of the overall system, composed of (1) and (10), is written as
\[
\dot{x} = -kLx + \text{diag}(a_1(t), \tilde{A}(t))x + [b_1(t), \tilde{B}^T(t)]^T, \quad (11)
\]
where \( x = [x_1, \ldots, x_N]^T \), \( \tilde{A}(t) = \text{diag}(a_2(t), \ldots, a_N(t)) \), \( \tilde{B}(t) = [b_2(t), \ldots, b_N(t)]^T \), and \( L \) is the Laplacian matrix. Since the system (11) satisfies global Lipschitz condition, there exists a unique solution for all \( t \geq t_0 \).

Consider the coordinate transformation \( \xi = U^{-1}x \) such that \( U^{-1}LU = \text{diag}(0, \Lambda) \) where \( \xi = [\xi_1, \xi^T]^T \), \( \xi = [\xi_2, \ldots, \xi_N]^T \), and \( \Lambda = \text{diag}(\lambda_2, \ldots, \lambda_N) \).

\[
\dot{\xi}_1 = \left\{ r_{11}a_1(t) + r_{12}\tilde{A}(t)1_{N-1}\right\}\xi_1 + \left\{ r_{11}a_1(t)Q_{12} + r_{12}\tilde{A}(t)Q_{22}\right\}\xi + \left\{ r_{11}b_1(t) + r_{12}\tilde{B}(t)\right\} \tag{12a}
\]

\[
\dot{\xi} = -\{kA - a_1(t)R_{21} - r_{22}\tilde{A}(t)Q_{22}\}\xi + \{a_1(t)R_{21} + r_{22}\tilde{A}(t)1_{N-1}\}\xi_1 + b_1(t)R_{21} + r_{22}\tilde{B}(t) \tag{12b}
\]

The system (12) can be considered as an interconnected system. From the coordinate transformation \( x = UC\), and the network property (2), it follows that the system (1) achieve AC if and only if there exists a diffusive coupling gain \( k \) such that \( \lim_{t \to \infty} \|\xi(t)\| = 0 \). Indeed, let \( \tilde{x} = [\tilde{x}_2, \ldots, x_N]^T \), then from the coordinate transformation,

\[
\begin{bmatrix} \xi_1 \\ \xi \end{bmatrix} = U^{-1}x = \begin{bmatrix} r_{11} & r_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ \tilde{x} \end{bmatrix} \\
\xi = R_{21}x_1 + R_{22}\tilde{x}, \tag{13}
\]

Thus, from the network property (2), if AC is achieved, then \( \xi \to 0 \). Conversely, let \( e_i \) is an \( N \times 1 \) matrix whose \( i \)-th entry is one and the others are zeros, and \( Q = [Q_{12}, Q_{22}]^T \). Then the \( i \)-th agent state can be described as \( x_i = c_i^T(1_N\xi_1 + Q\xi) \). Thus,

\[
x_i - x_j = (c_i^T - c_j^T)(1_N\xi_1 + Q\xi) = (c_i^T - c_j^T)Q\xi. \tag{14}
\]

Therefore, if there exists a diffusive coupling gain \( k \) such that \( \xi \to 0 \), then the system (1) achieves AC. From this fact, AC problems of MAS can be considered as the asymptotic stabilization of \( \xi \). However, because the system (12b) is an interconnected system, it is not easy to see that \( \|\xi\| \to 0 \) as \( t \to \infty \). Regarding the interconnected system (12b) as a nonvanishing perturbation of the isolated system, it is reasonable to expect the ultimate boundedness of \( \xi \). Similar to AC, we can say that PC problems of multi-agent systems can be considered as the problem of ultimate boundedness of \( \xi \).

**Lemma 1:** The system (1) achieves PC if, for any given \( \epsilon > 0 \), there exists a diffusive coupling gain \( k \) (dependent on \( \epsilon \)) and \( T \geq 0 \) (dependent on \( \epsilon \) and \( \xi(t_0) \)) such that \( \|\xi(t)\| \leq \frac{\epsilon}{\sqrt{2}\|Q\|} \), \( \forall t \geq t_0 + T \), where \( Q = [Q_{12}, Q_{22}]^T \).

**Proof:** From the equality (13), we obtain \( \|x_i - x_j\| \leq \|c_i^T - c_j^T\|\|Q\|\|\xi\| = \frac{\sqrt{2}\|Q\|\|\xi\|}{\sqrt{2}\|Q\|} \). Thus, if for any given \( \epsilon > 0 \), there exist a diffusive coupling gain \( k \) (dependent on \( \epsilon \)) and \( T \geq 0 \) (dependent on \( \epsilon \) and \( \xi(t_0) \)) such that \( \|\xi(t)\| \leq \frac{\epsilon}{\sqrt{2}\|Q\|} \), \( \forall t \geq t_0 + T \), then there exists a control input (10) and \( T \geq 0 \) such that \( \|x_i(t) - x_j(t)\| \leq \epsilon \), \( \forall t \geq t_0 + T \). It means that the system (1) achieves PC.

According to Definition 1 and 2, if AC is achieved, then so is PC.

### 3. Homogeneous Linear Time-Varying Multi-Agent Systems with Disturbances

In this section, we consider the case where the system (1) is homogeneous first-order LTV MAS with disturbances; i.e., \( a_i(t) = a(t) \) for all \( i \in N \) but \( b_i(t) \)'s are not necessarily identical.

**Assumption 1:** Suppose that there exists a constant \( M \) such that \( |a_i(t)| \leq M \), \( \forall t \geq t_0, \forall i \in N \).

Let us assume that the differences between disturbances are bounded.

**Assumption 2:** Suppose that there exists a constant \( D \) such that \( |b_i(t) - b_j(t)| \leq D \), \( \forall t \geq t_0, \forall i, j \in N \).

Since \( a_i(t) = a(t) \) for all \( i \in N \) and from the network properties (2) and (4), the system (12b) can be rewritten as

\[
\dot{\xi} = -\{kA - a(t)I_{N-1}\}\xi + b_1(t)R_{21} + r_{22}\tilde{B}(t). \tag{15}
\]

Note that the system (14) is no longer an interconnected systems.

**Theorem 1:** Under Assumption 1 and 2, the system (1) achieves PC if, for any given \( \epsilon > 0 \), there exists a diffusive coupling gain \( k \) such that

\[
k > \frac{\sqrt{2}D\|Q\| + \epsilon \theta M}{\theta \lambda_2}, \quad 0 < \theta < 1, \tag{16}
\]

where \( D = \left[ \sum_{i=2}^{N} \left( \sum_{j=1}^{N} |r_{ij}| \right)^2 \right]^{\frac{1}{2}} \), and \( \theta \) is the performance parameter.

**Proof:** From the network property (2), we can obtain \( r_{ii} = -\sum_{j=2}^{N} r_{ij} \) for \( 2 \leq i \leq N \). Thus, by Assumption 2, for \( 2 \leq i \leq N \), \( \sum_{j=1}^{N} r_{ij}b_j(t) = -\sum_{j=2}^{N} r_{ij}b_j(t) \). Thus, for any given \( \epsilon > 0 \), there exist a diffusive coupling gain \( k \) (dependent on \( \epsilon \)) and \( T \geq 0 \) (dependent on \( \epsilon \) and \( \xi(t_0) \)) such that \( \|\xi(t)\| \leq \frac{\epsilon}{\sqrt{2}\|Q\|} \), \( \forall t \geq t_0 + T \), then there exists a control input (10) and \( T \geq 0 \) such that \( \|x_i(t) - x_j(t)\| \leq \epsilon \), \( \forall t \geq t_0 + T \). It means that the system (1) achieves PC.

According to Definition 1 and 2, if AC is achieved, then so is PC.
Thus, by [13, Theorem 4.18], if $k > \frac{M}{\sqrt{2}}$, then $\xi$ is ultimately bounded by $\|\xi(t)\| \leq \frac{D B}{(s\sqrt{2} - Mt)}$, $\forall t \geq t_0 + T$, where $T \geq 0$ is depends on $\xi(t_0)$ and the ultimate bound. Therefore, since for any given $\epsilon > 0$, if $k$ satisfies (15),

$$\|\xi(t)\| \leq \frac{\epsilon}{\sqrt{2}\|Q\|}, \forall t \geq t_0 + T.$$ 

The system (1) achieves PC by Lemma 1.

In the case of homogeneous LTV MAS with homogeneous time-varying disturbances, we can let $D = 0$ in Assumption 2. Then, $\hat{\dot{V}} \leq -(k\lambda_2 - M)\|\xi\|^2$, $\forall \|\xi\| \in \mathbb{R}^{(N-1) \times 1}$. It means that the system (1) achieves AC.

**Corollary 1:** The system (1) achieves AC if $b_i(t) = b(t)$ for all $i \in \mathcal{N}$, and the diffusive coupling gain $k > \frac{M}{\sqrt{2}}$. □

If we consider that the system (14) is a perturbed system with vanishing perturbation, the system (1) achieves AC. That is, when $b_i(t)$’s approach each other as time goes to infinity, we obtain the following result from the statement 3 of [13, Lemma 9.6].

**Corollary 2:** The system (1) achieves AC if $\lim_{t \to \infty} |b_i(t) - b_j(t)| = 0$, $\forall i, j \in \mathcal{N}$, and the diffusive coupling gain $k > \frac{M}{\sqrt{2}}$. □

**Remark 2:** Since Assumption 2 includes the case where $b_i(t)$’s are not bounded, the above theorem and corollaries are not covered by Theorem 1 in [16]. □

4. HETEROGENEOUS LINEAR TIME-VARYING MULTI-AGENT SYSTEMS WITH DISTURBANCES

In this section, we consider the case where the system (1) is heterogeneous first-order LTV MAS with disturbances. That is, it is not necessary to be satisfied that $a_1(t) = \cdots = a_N(t)$ and $b_1(t) = \cdots = b_N(t)$. Note that the system (12) is an interconnected system. In order to simplify the analysis, we consider the interconnection term as the perturbation of the nominal subsystem. Because the perturbation term of the system dynamics $\xi$ depends on $\xi_1$ and $b_1(t)$’s, we need the following assumptions.

**Assumption 3:** Suppose there exists a constant $p > 0$ such that $\sum_{i=1}^{N} a_i(t) \leq -p$, $\forall t \geq t_0$. □

**Assumption 4:** Suppose there exists a constant $B$ such that $|b_i(t)| \leq B$, $\forall t \geq t_0, \forall i \in \mathcal{N}$. □

First, we consider the average system dynamics of all MAS

$$\dot{s} = \frac{1}{N} \sum_{j=1}^{N} \{a_j(t) s + b_j(t)\} = \{r_{11} a_1(t) + R_{12} \dot{\hat{A}}(t) 1_{N-1}\} s + r_{11} b_1(t) + R_{12} \dot{\hat{B}}(t), \quad (\cdot: \text{ (8)})\text{.}$$

Using Assumption 3 and 4, we show that $s(t)$ is ultimately bounded. By taking $V_s(s) = \frac{1}{2}s^2$ as a Lyapunov function, the derivative of $V_s$ along the trajectories of (16) is

$$\dot{V}_s = \{r_{11} a_1(t) + R_{12} \dot{\hat{A}}(t) 1_{N-1}\} s^2 + \{r_{11} b_1(t) + R_{12} \dot{\hat{B}}(t)\} s$$

$$\leq -r_{11} ps^2 + \|r_{11} b_1(t) + R_{12} \dot{\hat{B}}(t)\| |s|$$

$$\leq -r_{11} ps^2 + r_{11} N B|s|$$

$$= -r_{11} (1 - \theta_s) ps^2 - r_{11} \theta_s ps^2 + r_{11} N B|s|$$

$$\leq -r_{11} (1 - \theta_s) ps^2, \quad |s| \geq \frac{N B}{\theta_s p},$$

where $r_{11} = \frac{1}{\sqrt{2}} > 0$, and $0 < \theta_s < 1$. Therefore, by [13, Theorem 4.18], $s(t)$ is ultimately bounded by

$$|s(t)| \leq \frac{N B}{\theta_s p} =: B_s(\theta_s) \quad \forall t \geq t_0 + T_s, \quad (17)$$

where $T_s \geq 0$ depends on $s(t_0)$ and $B_s(\theta_s)$. Now we show that the system (1) achieves PC. Analogous to Theorem 1 in Section II, we present it by proving the ultimate boundedness of $\xi$. The proof is, however, not as simple as that of Theorem 1. Thus, we first need the following useful lemma.

**Lemma 2:** Consider the vector $Y = [a\|u\| + b\|v\|, \|u\|, \|v\|]^T$, where $a \neq 0$ and $b$ are real constants, $u$ and $v$ are vector variables. Then, the following inequality holds:

$$\|u\|^2 + \|v\|^2 \leq \max\left\{\frac{2}{a^2}, \frac{a^2 + 2b^2}{a^2}\right\} \|Y\|^2.$$ □

**Proof:** From the definition of Euclidian norm,

$$\|Y\|^2 = (a\|u\| + b\|v\|)^2 + \|v\|^2.$$

Let $x = a\|u\| + b\|v\|$, then $\|u\| = \frac{1}{a} x - \frac{b}{a} \|v\|$, and

$$\|u\|^2 + \|v\|^2 = \left(\frac{1}{a} x - \frac{b}{a}\right)^2 + \|v\|^2$$

$$\leq \frac{2}{a^2} x^2 + \frac{2b^2}{a^2} \|v\|^2 + \|v\|^2$$

$$\leq \max\left\{\frac{2}{a^2}, \frac{a^2 + 2b^2}{a^2}\right\} (x^2 + \|v\|^2)$$

$$= \max\left\{\frac{2}{a^2}, \frac{a^2 + 2b^2}{a^2}\right\} \|Y\|^2.$$

Now we present the main theorem of this section.

**Theorem 2:** Under Assumption 1, 3 and 4, the system (1) achieves PC if, for any given $\epsilon > 0$, there exists a diffusive coupling gain $k$ such that

$$k > \max\left\{k_1(\theta), k_2(\epsilon, \theta, \Theta_s), k_3(\Theta_s)\right\},$$

where $k_1(\theta) := \frac{N(N-1)M^2}{\theta^2 (\hat{Q}_2 ||Q_2|| + \|B_{22}\|)^2} + O$, \( k_2(\epsilon, \theta, \Theta_s) := \frac{2N\|Q_2\|^2 (M\sqrt{N-1} B_s(\theta_s) + B(\sqrt{N-1} + 1))^2}{\theta^2 \lambda_2 \epsilon^2} + O, \)

$$k_3(\Theta_s) := \frac{128N\|Q_2\|^2 (M\sqrt{N-1} B_s(\theta_s) + B(\sqrt{N-1} + 1))^2}{\theta^2 \lambda_2 \epsilon^2} + O,$$
ward calculations, it follows that
\[ O := \frac{NM||Q_{22}|| + ||R_{22}||}{\theta}B_x(\theta_s), \quad B_x(\theta_s) = \frac{NM}{\theta}, 0 < \theta < 1, \text{ and} \quad 0 < \theta_s < 1. \]

\[ \boxed{\text{Proof: Let } e = \xi_1 - s, \text{ then, by (12) and (16),}} \]
\[ \dot{\xi} = \{r_{11}a_1(t) + R_{12}\tilde{A}(t)1_{N-1}\}e + \{r_{11}a_1(t)Q_{12} + R_{12}\tilde{A}(t)Q_{22}\}\xi. \]  
(18)
\[ \dot{\xi} = \{kA - a_1(t)R_{21}Q_{12} - R_{22}\tilde{A}(t)Q_{22}\}\xi + \{a_1(t)R_{21} + R_{22}\tilde{A}(t)1_{N-1}\}e + \{a_1(t)Q_{12} + R_{22}\tilde{A}(t)1_{N-1}\}s + b_1(t)R_{21} + R_{22}\tilde{B}(t). \]  
(19)

By taking \( V_1(e) = \frac{1}{2}e^2 \) and using (18), the derivative of \( V_1 \) along the trajectories of (18) is
\[ \dot{V}_1 = \{r_{11}a_1(t) + R_{12}\tilde{A}(t)1_{N-1}\}e^2 + e\{r_{11}a_1(t)Q_{12} + R_{12}\tilde{A}(t)Q_{22}\}\xi. \]  
(20)

Also, by taking \( V_2(\xi) = \frac{1}{2}\xi^T\tilde{\xi} \) and using (19), the derivative of \( V_2 \) along the trajectories of (19) is
\[ \dot{V}_2 = -\xi^T\{kA - a_1(t)R_{21}Q_{12} - R_{22}\tilde{A}(t)Q_{22}\}\xi + e\{a_1(t)R_{21} + 1_{N-1}\tilde{A}(t)R_{21}\}\xi + s\{a_1(t)R_{21} + 1_{N-1}\tilde{A}(t)R_{21}\}b_1(t)R_{21} + \tilde{B}(t)^TR_{21}\xi. \]  
(21)

We consider the function \( V = V_1 + V_2 \) as a composite Lyapunov function. By using (20) and (21), the derivative of \( V \) is
\[ \dot{V} = \{r_{11}a_1(t) + R_{12}\tilde{A}(t)1_{N-1}\}e^2 + e\{r_{11}a_1(t)Q_{12} + R_{12}\tilde{A}(t)Q_{22} + a_1(t)R_{21} + 1_{N-1}\tilde{A}(t)R_{21}\}\xi - \tilde{\xi}^T\{kA - a_1(t)R_{21}Q_{12} - R_{22}\tilde{A}(t)Q_{22}\}\xi + \{s\{a_1(t)R_{21} + 1_{N-1}\tilde{A}(t)R_{21}\}b_1(t)R_{21} + \tilde{B}(t)^TR_{21}\xi. \]

From (6)-(9), (17), Assumption 1, 3, 4, and straightforward calculations, it follows that
\[ \dot{V} \leq \frac{-p}{N}e^2 + 2M\sqrt{N-1}\left(\frac{1}{N}||Q_{22}|| + ||R_{22}||\right)||e|| ||\xi|| + (2M\sqrt{N-1}B_x(\theta_s) + B(\sqrt{N-1} + 1))||R_{22}|| ||\xi|| - (k\lambda_2 - NM||R_{22}|| ||Q_{22}|| ||\xi||) ||e||, \quad \forall t \geq t_0 + T_s, \]
\[ = \frac{-p}{N}e^2 + 2M\sqrt{N-1}\left(\frac{1}{N}||Q_{22}|| + ||R_{22}||\right)||e|| ||\xi|| + (2M\sqrt{N-1}B_x(\theta_s) + B(\sqrt{N-1} + 1))||R_{22}|| ||\xi|| - (1 - \theta)(k\lambda_2 - NM||R_{22}|| ||Q_{22}|| ||\xi||) ||e||^2 - \theta(k\lambda_2 - NM||R_{22}|| ||Q_{22}|| ||\xi||) ||\xi||^2. \]

If \( k > k_1(\theta) \), then
\[ \dot{V} \leq -\frac{p}{N}e^2 + 2M\sqrt{N-1}\left(\frac{1}{N}||Q_{22}|| + ||R_{22}||\right)||e|| ||\xi|| - \frac{N(N-1)M^2\left(\frac{1}{N}||Q_{22}|| + ||R_{22}||\right)^2}{p} ||\xi||^2 - \theta(k\lambda_2 - NM||R_{22}|| ||Q_{22}|| ||\xi||) ||e|| \]
\[ + (2M\sqrt{N-1}B_x(\theta_s) + B(\sqrt{N-1} + 1))||R_{22}|| ||\xi|| \]
\[ = -\left\{ \frac{p}{N}||e|| - \frac{M\sqrt{N-1}(\frac{1}{N}||Q_{22}|| + ||R_{22}||)}{\sqrt{N}} ||\xi||^2 \right\} + (2M\sqrt{N-1}B_x(\theta_s) + B(\sqrt{N-1} + 1))||R_{22}|| ||\xi|| - \theta(k\lambda_2 - NM||R_{22}|| ||Q_{22}|| ||\xi||) ||e||. \]

Let
\[ \delta(k, \theta) := \sqrt{\theta(k\lambda_2 - NM||R_{22}|| ||Q_{22}||)}, \]
\[ W(\theta_s) := 2M\sqrt{N-1}B_x(\theta_s) + B(\sqrt{N-1} + 1), \]
and
\[ Y := \left[ \begin{array}{c} \frac{\sqrt{p}}{N}||e|| - \frac{M\sqrt{N-1}(\frac{1}{N}||Q_{22}|| + ||R_{22}||)}{\sqrt{N}} ||\xi|| \end{array} \right] - \frac{\sqrt{p}}{2\sqrt{N} \delta(k, \theta)} \right] \]

then inequality of derivative of \( V \) is rewritten as
\[ \dot{V} \leq - (\delta(k, \theta))^2 ||Y||^2 - W(\theta_s)||Y||. \]

For any given \( \epsilon > 0 \), let \( \rho(\epsilon, k, \theta) := \frac{\sqrt{\epsilon \rho \rho}}{2\sqrt{N} \delta(k, \theta)} \) where \( \epsilon := \frac{\sqrt{\rho \rho}}{2\sqrt{N} \delta(k, \theta)} \). If \( k > k_2(\theta, \theta, \theta) \), then \( W(\theta_s) < \frac{\sqrt{\rho \rho}}{4\sqrt{N}} \delta(k, \theta) \). Thus, if \( ||Y|| \geq \rho(\epsilon, k, \theta) \), then
\[ \dot{V} \leq ||Y||^2 \left\{ - (\delta(k, \theta))^2 ||Y||^2 + W(\theta_s) \right\} \]
\[ \leq \frac{\sqrt{\epsilon \rho \rho}}{2\sqrt{N} \delta(k, \theta)} \left\{ \frac{\sqrt{\rho \rho}}{2\sqrt{N} \delta(k, \theta)} \right\} \]
\[ = -\frac{1}{8N}p\epsilon^2 < 0, \quad \forall t \geq t_0 + T_s. \]

Also, if \( ||Y|| \leq \rho(\epsilon, k, \theta) \), then from Lemma 2
\[ V(e, \xi) = \frac{1}{2}e^2 + \frac{1}{2}\xi^T\xi = \frac{1}{2}||e||^2 + \frac{1}{2}||\xi||^2 \]
\[ \leq \max \left\{ \frac{1}{p^2}, 1 + \frac{2M^2N(N-1)(\frac{1}{N}||Q_{22}|| + ||R_{22}||)}{\rho^2} \right\} (\rho(\epsilon, k, \theta))^2. \]

If \( k \geq k_3(\theta) \), then
\[ V(e, \xi) \leq \frac{2N(\delta(k, \theta))^2}{p^2} \left( \frac{\sqrt{\rho \rho}}{2\sqrt{N} \delta(k, \theta)} \right)^2 \leq \frac{1}{2}. \]

Because the system (11) satisfies a global Lipschitz condition, there is no finite escape time over \( [t_0, t_0 + T_s] \). Now, we consider the following two cases after \( t = t_0 + T_s \).
• Case 1) When $t = t_0 + T_s$, $\|Y\| < \rho(\epsilon, k, \theta)$: In this case, the inequality (23) is satisfied when $t = t_0 + T_s$. Even if there exists $T_1 \geq 0$ such that $\|Y\|_2 = \rho(\epsilon, k, \theta)$ when $t = t_0 + T_s + T_1$, the inequality (23) is still satisfied. Furthermore, if $\|Y\| \geq \rho(\epsilon, k, \theta)$, then inequality (22) is satisfied. Thus, for $t \geq t_0 + T_s + T_1$, the Lyapunov function $V(e(t), \tilde{t}(t))$ cannot be larger than $V(e(t_0 + T_s + T_1), \tilde{t}(t_0 + T_s + T_1))$. Therefore, for all $t \geq t_0 + T_s + T_1$, $V(e(t), \tilde{t}(t)) \leq V(e(t_0 + T_s + T_1), \tilde{t}(t_0 + T_s + T_1)) \leq \epsilon^2$.

• Case 2) When $t = t_0 + T_s$, $\|Y\| \geq \rho(\epsilon, k, \theta)$. In this case, there exists $T_2 \geq 0$ such that $\|Y\| < \rho(\epsilon, k, \theta)$ when $t = t_0 + T_s + T_2$. If not, then it means that for all $t \geq t_0 + T_s$, the inequality $\|Y\| \geq \rho(\epsilon, k, \theta)$ is satisfied. However, from the inequality (22),

$$V(e(t), \tilde{t}(t)) \leq -\frac{1}{8} r_{11} p \epsilon^2 (t - (t_0 + T_s)) + V(e(t_0 + T_s), \tilde{t}(t_0 + T_s)),$$

for all $t \geq t_0 + T_s$. If $t > t_0 + T_s + \frac{V(e(t_0 + T_s), \tilde{t}(t_0 + T_s))}{\frac{1}{8} r_{11} p \epsilon^2}$, then $V(e(t), \tilde{t}(t)) < 0$ and this is a contradiction to the fact that $V(e, \tilde{t}) \geq 0$ for all $e, \tilde{t}$. Thus, there exists $T_2 > 0$ such that $\|Y\| < \rho(\epsilon, k, \theta)$ when $t = t_0 + T_s + T_2$. It is connected to Case 1).

From the above Case 1) and Case 2), for any give $\epsilon > 0$, if $k > \max \left\{k_1(\theta), k_2(\epsilon, \theta, \theta_3), k_3(\theta) \right\}$, where $0 < \theta < 1$, $0 < \theta_3 < 1$, then

$$V(e, \tilde{t}) \leq \frac{1}{2} \epsilon \xi^2 + \frac{1}{2} \tilde{t}^T \xi \leq \frac{1}{2} \epsilon \xi^2 + \frac{1}{2} \tilde{t}^T \xi \leq \frac{1}{2} \epsilon \xi^2 \leq \frac{1}{2} \epsilon \xi^2, \forall t \geq t_0 + T_s + T_2.$$

It means that $\|\tilde{t}(t)\| \leq \epsilon = \frac{1}{\sqrt{2q(s)}}$ for all $t \geq t_0 + T_s + T_2$. Therefore, by Lemma 1, we can say that the system (1) achieves PC.

5. CONCLUSION

We have studied the PC problem for heterogeneous LTV MAS under a fixed, connected, and undirected communication network. We have provided sufficient conditions that, PC is achieved if the diffusive coupling gain is larger than the lower bound which has also been presented in this paper. Since the results have been given for heterogeneous LTV MAS, it can be easily applied to rather restricted dynamics such as homogeneous LTV or heterogeneous LTI MAS.

REFERENCES


