On Robustness of Synchronization in Heterogeneous Multi-Agent Systems

Jaeyong Kim, Jongwook Yang, Hyungbo Shim,* and Jung-Su Kim

Abstract—This paper studies robustness of synchronization in heterogeneous multi-agent systems, which is gained by interactions with other agents through the network. In order to effectively deal with the heterogeneous cases, we introduce the concept of the averaged dynamics which is the average of all agents’ dynamics, and then claim that two sources enhance the robustness of the group behavior against noise and perturbations. First, we show that strong coupling among agents makes the solutions of all agents remain in an arbitrary neighborhood of the trajectory of the averaged dynamics. Second, we illustrate by simulation analysis that as the number of agents increases in the network, the averaged dynamics with noise or perturbations approaches its nominal one that is the averaged dynamics without noise and perturbations. Finally, we also illustrate that strong coupling and a large number of agents yield that every agent’s dynamics synchronizes with the nominal averaged dynamics.

I. INTRODUCTION

We study the behavior of a group of \( N \) dynamic agents represented by

\[
\dot{x}_i = f_i(t, x_i) + u_i, \quad i \in \mathcal{N} = \{1, 2, \ldots, N\}
\]

(1)

where \( x_i \in \mathbb{R} \) is the state\(^1\) and \( u_i \in \mathbb{R} \) indicates interactions with other agents through the network. Here the function \( f_i(t, x_i) \) also includes possible time-varying disturbances entering each agent as well as parametric variations or uncertainties of each agent. And it is claimed that, under the interaction given by the diffusive-type coupling [1]

\[
u_i = k \sum_{j=1}^{N} \alpha_{ij} (x_j - x_i)
\]

(2)

where \( k \) represents the coupling strength and \( \alpha_{ij} \) is the \((i, j)\)-th entry of the adjacency matrix of the given network, the robustness of the group behavior is enhanced. By the group behavior, we mean the solution \( s(t) \) of the following averaged dynamics:

\[
\dot{s}(t) = \frac{1}{N} \sum_{i=1}^{N} f_i(t, s(t))
\]

(3)

with the averaged initial condition \( s(0) = \frac{1}{N} \sum_{i=1}^{N} x_i(0) \). It will be shown that each solution \( x_i(t) \) satisfies that

\[
\lim_{t \to \infty} \sup_{k > \tilde{K}} |x_i(t) - s(t)| \leq \sigma(1/k), \quad \forall k > \tilde{K}
\]

(4)

where \( \tilde{K} \) is a minimal required strength and \( \sigma \) is a class-\(\mathcal{K}\) function.\(^2\)

Then, the robustness of the group behavior is caused by two sources: strong coupling and a large number of agents. First, even though the dynamics of each agent are all different, or uncertain, their behaviors approach close to that of \( s(t) \) when the coupling strength \( k \) is large. It is therefore interpreted that strong coupling enhances the robustness of the group behavior, as seen in (4). Second, as the number of agents \( N \) increases in the network, the averaged dynamics (3) with perturbations and noise approaches the nominal averaged dynamics without them. For example, if \( f_i(t, x_i) = \alpha_i x_i + \Delta_i \), the averaged dynamics becomes

\[
\dot{s} = \left( \frac{1}{N} \sum_{i=1}^{N} \alpha_i \right) s + \frac{1}{N} \sum_{i=1}^{N} \Delta_i
\]

where \( \Delta_i \)'s are independent and identically distributed (i.i.d.) random variables with the standard normal distribution \( N(0, 1) \). When \( N \) is sufficiently large, the effects of \( \Delta_i \)'s are weakened in the averaged dynamics; i.e., the standard deviation of \( \frac{1}{N} \sum_{i=1}^{N} \Delta_i \) is \( \frac{1}{\sqrt{N}} \), so it converges to zero as \( N \) goes to infinity.

In the next subsection, some motivation of the studied problem is presented. Then, in Section II, we present a theorem on the robustness by strong coupling. Section III discusses the robustness by a large number of agents, and also illustrate that strong coupling and a large number of agents yield that every agent’s dynamics synchronizes with the nominal averaged dynamics. Finally, Section IV concludes the paper.

A. Motivation of the Study

During the last decade, synchronization in engineering and nature has received considerable interests. This is because it turns out that there are several dynamical properties of synchronized dynamics which are not found in dynamics of an individual system [2], [3]. For instance, the circadian oscillator is the representative of this. Many life phenomena in biological systems are heavily dependent on the time of day. In order to provide these biological systems with robust

\(^1\)In this paper, we deal with only the scalar systems. However, we believe the proposed treatment can also be extended to higher order dynamics, which will be pursued in the near future.

\(^2\)A continuous function \( \sigma : [0, a) \to [0, \infty) \) is said to belong to class-\(\mathcal{K}\) if it is strictly increasing and \( \sigma(0) = 0 \).

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As mentioned above, an example of remarkable properties of synchronized dynamics is robustness. It is reported that synchronized dynamics via coupling between oscillators enhance robustness of oscillating dynamics against noise or parameter variations; see [5]–[7] and references therein. This robustness implies that the oscillating behavior under noise or parameter variations is as close as possible to the nominal oscillating behavior (protected from them). This means that although an individual oscillating behavior is not robust under the uncertainties, collective oscillators can be indeed so by achieving synchronization via coupling between them.

This paper focuses on the question: what is the most general principle behind this finding? To be more specific, this finding might hint at (or motivates to make) the hypothesis that a large-scale synchronization-like behavior shows robustness against noise. As already mentioned, the noise can be attenuated in the synchronization-like behavior if the number of agents $N$ is sufficiently large. Thus, in this paper, we will consider the interconnected systems to be robust against noise if their behaviors are close to their averaged dynamics without noise.

In the case of the identical agents with noise, both strong coupling and a large number of agents are required to achieve the noise-free synchronization [5], [7]. Under these preconditions (strong coupling and a large number of agents), the behavior of the identical individual dynamics is arbitrary close to that of a noise-free system [5]. However, the behavior of complex networks with nonidentical systems is much more complicated than that of the identical case [15]. In this paper, we introduce the averaged dynamics (3) which is the average not of the states, but of the dynamics, in order to effectively describe the group behavior of nonidentical subsystems. Then, we will show that strong coupling guarantees the synchronization with the averaged dynamics, and this will show that strong coupling contributes to the robustness. Furthermore, as seen in the above example (i.e., $f_i(t, x_i) = a_i x_i + \Delta_i$), the noise attenuation effect of a large number of agents is better understood in the averaged dynamics.

On the other hand, the considered problem can be viewed as achieving the practical consensus from the viewpoint of the consensus problems discussed in, e.g., [8]–[12]. The practical consensus problem can be defined as follows: for any given $\epsilon > 0$, design the consensus controller (2) for the multi-agent system (1) such that

$$\limsup_{t \to +\infty} |x_i(t) - x_j(t)| \leq \epsilon, \quad \forall i, j \in \mathcal{N}.$$  

This approximate consensus is the best we can achieve because (1) is the heterogeneous multi-agent system for which exact (or asymptotic) consensus is not possible if there is no common internal model among them. See [13]–[15] for details. In particular, Zhao et al. [15] have considered a similar problem, but they assume that the averaged solution $s(t)$ of (3) is also a solution of each subsystem, i.e., $\dot{s}(t) = f_i(t, s(t))$, $i \in \mathcal{N}$. Unlike this, the averaged solution $s(t)$ may not be the solution to any subsystem (1) in this paper.

### B. Notation

A directed graph is denoted by $G = (\mathcal{N}, \mathcal{E}, A)$, where $\mathcal{N} = \{1, 2, \ldots, N\}$ is a finite nonempty set of nodes, $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ an edge set of ordered pairs of nodes, and $A = [a_{ij}] \in \mathbb{R}^{N \times N}$ an adjacency matrix. An edge $(i, j) \in \mathcal{E}$ implies that the information flows from the node $i$ to the node $j$. A graph is said to be undirected if it has the property that $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$ for any node $i, j \in \mathcal{N}$. We exclude the self-connection, i.e., $(i, i) \notin \mathcal{E}$. It is related with $\alpha_{ji}$ by the rule that $\alpha_{ji} = 1$ if and only if $(i, j) \in \mathcal{E}$.

Otherwise $\alpha_{ji} = 0$. A path (of length $l$) from node $i$ to node $j$ is a sequence $(i_0, i_1, \ldots, i_l)$ of nodes such that $i_0 = i$, $i_l = j$, $(i_k, i_{k+1}) \in \mathcal{E}$, and $i_k$'s are distinct. An undirected graph $\mathcal{G}$ is connected if there is a path from $i$ to $j$ for arbitrary two distinct nodes $i, j \in \mathcal{N}$.

The Laplacian $L = [l_{ij}] \in \mathbb{R}^{N \times N}$ of $\mathcal{G}$ is defined as $L := \mathcal{D} - \mathcal{A}$, where the $i$th diagonal of the diagonal matrix $\mathcal{D}$ is given by $d_i := \sum_{j \in \mathcal{N}} a_{ij}$. By its construction, it contains a zero eigenvalue with a corresponding eigenvector $1_N$ (i.e., $\mathcal{L}1_N = 0$) and all the other eigenvalues lie in the open right-half complex plane, where $1_N$ denotes the $N \times 1$ column vector comprising all ones. Thus, we sort them as $0 = \lambda_1 \leq \text{Re}(\lambda_2(\mathcal{L})) \leq \cdots \leq \text{Re}(\lambda_N(\mathcal{L}))$, where $\lambda_i$'s are eigenvalues of $\mathcal{L}$. The zero eigenvalue is simple if and only if the corresponding graph $\mathcal{G}$ is connected [10]. For undirected graphs, both $A$ and $\mathcal{L}$ are symmetric so that $\lambda_i$'s are real nonnegative numbers.

For matrices $A_1, \cdots, A_k$, $\text{diag}(A_1, \cdots, A_k)$ is defined as the block diagonal matrix whose the $i$th diagonal entry is $A_i$. For a vector $x$ and a matrix $A$, $\|x\|$ and $\|A\|$ denote the Euclidean norm and the induced matrix 2-norm, respectively.

### II. ROBUSTNESS BY STRONG COUPLING

We study the problem under the following assumptions.

**Assumption 1:** (Individual system) The function $f_i(t, x_i)$ of the individual system (1) is uniformly bounded in $t$, continuously differentiable, and globally Lipschitz in $x_i$ uniformly in $t$; i.e., there exist a non-decreasing continuous function $M : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a constant $L$ such that

$$|f_i(t, a)| \leq M(|a|), \quad \left| \frac{\partial f_i(t, x_i)}{\partial x_i} \right| \leq L, \quad \forall x_i \in \mathbb{R}, \quad \forall t \geq 0, \quad \forall i \in \mathcal{N}. \quad \Box$$

Assumption 1 guarantees the uniqueness of solution for the individual system. Moreover, if the domain of $f_i$ is restricted, a constant $L$ can always be found. This assumption makes the analysis simple.
By letting \( x := [x_1, \ldots, x_N]^T \) and \( f(t, x) := [f_1(t, x_1), \ldots, f_N(t, x_N)]^T \), the dynamics of the overall system, composed of (1) and (2), is written as

\[
\dot{x} = -kLx + f(t, x),
\]

where \( L \) is the Laplacian matrix describing the network connection.

**Assumption 2:** (Network property) The coupling network topology under consideration is undirected and connected.

All-to-all and ring network interconnections are particular kind of undirected and connected network topology. A direct consequence of the assumption is that the Laplacian matrix \( L \) is symmetric and has zero eigenvalue which is simple [10]. Therefore, by Schur’s lemma, there exists a normal (real diagonal matrix \( \Lambda \) with a real diagonal matrix \( \Lambda \in \mathbb{R}^{(N-1) \times (N-1)} \). From the property of Laplacian matrix of a connected symmetric graph, it follows that \( \Lambda = \text{diag}(\lambda_2, \lambda_3, \ldots, \lambda_N) \) with all positive \( \lambda_i \)’s, and without loss of generality, we suppose that the first row of \( U \) is \((1/\sqrt{N})[1, 1, \ldots, 1] \). Define the matrix \( W := (1/\sqrt{N})U \). Then,

\[
W = \left[ \frac{1}{\sqrt{N}} \right]^T R, \quad W^{-1} = [1, Q],
\]

where \( R \) and \( Q \) are real matrices of size \( N \times (N-1) \) such that \( R^T R = (1/N)I \), \( Q^T Q = NI \), \( R^T 1_N = 0 \), and \( Q^T 1_N = 0 \), and \( Q^T 1_N = 0 \). Hence, \( \|Q\| = \sqrt{N} \) and \( \|R\| = 1/\sqrt{N} \).

Now, for the averaged dynamics (3) which is simply written as

\[
\dot{s} = \frac{1}{N} 1_N f(t, 1_N s),
\]

the following properties are required in the analysis.

**Assumption 3:** (Averaged dynamics) (i) The solution \( s(t) \) from any initial condition is ultimately bounded in the sense that there exists \( B \) such that

\[
\limsup_{t \to \infty} |s(t)| = B.
\]

(ii) Average of Jacobians of individual systems is strictly negative. More specifically, there exists a constant \( p > 0 \) such that

\[
\frac{1}{N} \frac{\partial}{\partial x} (1_N f(t, x)) 1_N = \frac{1}{N} \sum_{i=1}^N \frac{\partial f_i}{\partial x_i}(t, x_i) \leq -p,
\]

\( \forall x_i \in \mathbb{R}, \ \forall t \geq 0 \).

**Remark 1:** The condition (9) is stronger than asking that the averaged dynamics (7) is a contracting system [16] since, from (9), it holds that \( (1/N)(\partial 1_N f(t, 1_N s))/\partial s \leq -p \) for all \( s \in \mathbb{R} \).

Under the assumptions so far, we obtain the following theorem.

**Theorem 1:** Under Assumptions 1, 2, and 3, there exists a class-K function \( \sigma^* \) such that the solutions of (1) with arbitrary initial conditions and the solution \( s(t) \) to the averaged dynamics (3) with \( s(0) = \frac{1}{N} \sum_{i=1}^N x_i(0) \) satisfy

\[
\limsup_{t \to \infty} |x_i(t) - s(t)| \leq \sigma^*(\frac{1}{k\lambda_2 - L}), \ \forall k > \tilde{K},
\]

for all \( i = 1, \ldots, N \), where

\[
\tilde{K} = \frac{L^2}{p\lambda_2} + \frac{L}{\lambda_2}.
\]

In particular, the function \( \sigma^* \) is defined on \([0, p/L^2] \) and given by

\[
\sigma^*(\chi) = M(B)\sqrt{N}\sqrt{r(\chi)}
\]

in which,

\[
r(\chi) = \begin{cases} 
0, & \chi = 0, \\
\frac{2\chi}{(\sigma^2 + 2L^2)\chi^2}, & 0 < \chi \leq \frac{2p}{p^2 + 4L^2}, \\
\frac{2p}{p^2 + 4L^2} < \chi < \frac{p}{L^2}.
\end{cases}
\]

**Remark 2:** The ultimate bound expressed by the function \( \sigma^* \) and the value of \( \tilde{K} \) may be conservative. However, the current expressions (11) and (12) yield a reasonable interpretation. For example, (11) indicates that the minimal coupling strength \( k \) increases as the degree of stability \( p \) and the smallest non-zero eigenvalue \( \lambda_2 \) of the network get larger.

**Remark 3:** Theorem 1 may be considered as a solution to the practical consensus problem (for the multi-agent system (1)), as discussed in the Introduction. In fact, for any given \( \varepsilon \), there is a sufficiently large \( k \) such that \( \limsup_{t \to \infty} |x_i(t) - x_j(t)| \leq \varepsilon \) for \( i, j \in \mathcal{N} \). Since the terminology ‘practical consensus’ is used differently in [17] where just boundedness of the difference \( |x_i(t) - x_j(t)| \) is of interest, we emphasize that the error could be made arbitrarily small in Theorem 1.

**A. Proof of Theorem 1**

By the coordinate transformation

\[
\xi = \begin{bmatrix} \xi_1 \\ \xi \end{bmatrix} = W x = \begin{bmatrix} \frac{1}{N} 1_N^T \end{bmatrix} x
\]

where \( \tilde{\xi} = [\xi_2, \ldots, \xi_N]^T \), the overall system (6) is transformed into

\[
\dot{\xi}_1 = \frac{1}{N} 1_N f(t, 1_N \xi_1 + \xi^*), \\
\dot{\xi} = \frac{1}{N} 1_N f(t, 1_N \xi_1 + \xi^*),
\]

because \( W^{-1} \xi = 1_N \xi_1 + \xi^* \). With \( e := \xi_1 - s \), equation (15) can be rewritten as

\[
\dot{e} = \frac{1}{N} 1_N f(t, 1_N e + 1_N s + \xi^*) - \frac{1}{N} 1_N f(t, 1_N s)
\]

\[
\dot{\xi} = -k\Lambda \xi + R^T f(t, 1_N e + 1_N s + \xi^*).
\]

Let a Lyapunov function be

\[
V(e, \xi) = \frac{1}{2} e^2 + \frac{1}{2} \xi^T \xi.
\]
Then, the time derivative of $V$ along (16) becomes

$$
\dot{V} = \frac{e}{N} \left[ 1_N^T f(t, 1_N e + 1_N s + Q \hat{\xi}) - 1_N^T f(t, 1_N s) \right] - k \xi^T \dot{\xi}^c + \frac{\partial (\xi^T R^T f)}{\partial x} \bigg|_{w} (1_N e + Q \hat{\xi}) - \xi^T R^T f(t, 1_N s) + \xi^T R^T f(t, 1_N s).
$$

By the mean-value theorem, we obtain

$$
\dot{V} = \frac{e}{N} \left[ 1_N^T f(t, 1_N e + 1_N s + Q \hat{\xi}) \right] - k \xi^T \dot{\xi}^c + \frac{\partial (\xi^T R^T f)}{\partial x} \bigg|_{w} (1_N e + Q \hat{\xi}) + \xi^T R^T f(t, 1_N s)
$$

in which, $z \in \mathbb{R}^N$ and $w \in \mathbb{R}^N$ are some points on the line segment connecting $1_N e + Q \hat{\xi} + 1_N s$ and $1_N s$. Since

$$
\frac{\partial (\xi^T R^T f)}{\partial x} \bigg|_{w} = \xi^T R^T \text{diag} \left( \frac{\partial f_1}{\partial x_1}(t, w_1), \cdots, \frac{\partial f_N}{\partial x_N}(t, w_N) \right),
$$

it is seen by (5) that

$$
\left| \frac{\partial (\xi^T R^T f)}{\partial x} \right| \bigg|_{w} = \leq L \|R\| \|\hat{\xi}\|, \quad \forall t \geq 0
$$

and similarly

$$
\left| \frac{\partial (1_N^T f)}{\partial x} \right| \bigg|_{z} \leq L \sqrt{N}, \quad \forall t \geq 0.
$$

Therefore, using (9) and the fact that $\|Q\| = \sqrt{N}$ and $\|R\| = 1/\sqrt{N}$, it follows that

$$
\dot{V} \leq -p\|e\|^2 + \frac{L\|Q\|}{\sqrt{N}} \|e\| \|\hat{\xi}\| - k \lambda_2 \|\hat{\xi}\|^2
$$

$$
+ L \|R\| \|\hat{\xi}\| (\sqrt{N}) \|e\| + \|Q\| \|\hat{\xi}\|) + |R^T f(t, 1_N s)| \|\hat{\xi}\|
$$

$$
\leq -p\|e\|^2 + 2L\|e\| \|\hat{\xi}\| - (k \lambda_2 - L) \|\hat{\xi}\|^2 + |R^T f(t, 1_N s)| \|\hat{\xi}\|.
$$

With $k_1 := k \lambda_2 - L$ and $a = -L$, the following lemma can be employed to find the region for $V < 0$.

Lemma 1: Let

$$
\rho_{k_1}(x, y) := \left[ \begin{array}{c} x \\ y \end{array} \right]^T \left[ \begin{array}{cc} p & a \\ a & k_1 \end{array} \right] \left[ \begin{array}{c} x \\ y \end{array} \right] + M \|y\|
$$

with $x \in \mathbb{R}^l$, $y \in \mathbb{R}^m$, $p > 0$, $M > 0$ and $a$ is a constant. Then, there is a class-$K$ function $r(\cdot)$ such that

$$
\rho_{k_1}(x, y) < 0 \quad \text{on } \{(x, y) : |x|^2 + |y|^2 > M^2 r(1/k_1)\},
$$

for all $k_1 > a^2/p$.

Proof: Note that

$$
\rho_{k_1}(x, y) = -p|x|^2 - 2a|x||y| - k_1|y|^2 + M|y|
$$

$$
= -p \left( |x| + \frac{a}{p} |y| \right)^2 - \left( k_1 - \frac{a^2}{p} \right) |y|^2 + M|y|.
$$

Let $\delta(k_1) := k_1 - a^2/p$ and $k_1 > a^2/p$ so that $\delta(k_1) > 0$, and let

$$
Y_{k_1}(x, y) := \left[ \begin{array}{c} \sqrt{p} \\ \sqrt{\delta(k_1)} \end{array} \right] \left[ \begin{array}{c} |x| + \frac{a}{p} |y| \\ |y| \end{array} \right]^T.
$$

Then, since $|y| \leq |Y_{k_1}(x, y)|$,

$$
\rho_{k_1}(x, y) = -\delta(k_1) \left[ \frac{p}{\delta(k_1)} \left( |x| + \frac{a}{p} |y| \right)^2 + |y|^2 \right] + M|y|
$$

$$
\geq -\delta(k_1) |Y_{k_1}(x, y)|^2 + M|y|.
$$

Therefore, $\rho_{k_1}(x, y) < 0$ if $|Y_{k_1}(x, y)| > M/\delta(k_1)$.

For convenience, let

$$
Z := \frac{\sqrt{p}}{\delta(k_1)} (|x| + \frac{a}{p} |y|).
$$

Then, we have

$$
|x|^2 + |y|^2 = \left( \frac{\sqrt{\delta(k_1)}}{\sqrt{p}} Z - \frac{a}{p} |y| \right)^2 + |y|^2
$$

$$
\leq \frac{2\delta(k_1)}{p} Z^2 + \frac{2a^2}{p^2} |y|^2 + |y|^2
$$

$$
\leq \eta(k_1) |Y_{k_1}(x, y)|^2,
$$

where $\eta(k_1) = \max \left\{ 2\delta(k_1)/p, 1 + 2a^2/p^2 \right\}$. Define $r : [0, p/a^2) \to [0, \infty)$ as follows:

$$
r(\chi) := \begin{cases} 
0 & \text{if } \chi = 0, \\
\sup_{\mu \geq \frac{a}{2}} \frac{\eta(\mu)}{\delta(\mu)} & \text{if } 0 < \chi < \frac{a^2}{p^2}.
\end{cases}
$$

(If $a = 0$, the number $p/a^2$ is replaced by $\infty$.) Then, it can be verified that the function $r(\chi)$ belongs to class-$K$ because it is equivalently written as (13).

Finally, if $|x|^2 + |y|^2 > M^2 r(1/k_1)$, then

$$
|Y_{k_1}(x, y)|^2 \geq \frac{1}{\eta(k_1)} (|x|^2 + |y|^2) > M^2 \eta(k_1) r(1/k_1)
$$

$$
\geq M^2 \eta(k_1) \delta(k_1) = \left( \frac{M}{\Delta(k_1)} \right)^2,
$$

and thus $\rho_{k_1}(x, y) < 0$. This completes the proof. $\blacksquare$

By Lemma 1, it is seen that

$$
\dot{V} < 0 \quad \text{if } 2V < e^2 + |\xi|^2 \leq |R^T f(t, 1_N s)|^2 r(1/k_1),
$$

which implies that

$$
\limsup_{t \to \infty} 2V(t) \leq \limsup_{t \to \infty} |R^T f(t, 1_N s)|^2 r(1/k_1).
$$

By (5) and (8), we have that

$$
\limsup_{t \to \infty} |R^T f(t, 1_N s(t))| \leq \|R\| \sqrt{N} M \text{lim sup} |s(t)|
$$

$$
= M(B).
$$

Finally, note that

$$
x_1 - 1_N s = W^{-1} \xi_1 - 1_N s_1 = 1_N \xi_1 - 1_N s + Q \hat{\xi} = [1_N, Q] \left[ \begin{array}{c} e \\ \hat{\xi} \end{array} \right].
$$

Then, the vector norm of the $i$-th row of $[1_N, Q]$ is $\sqrt{N}$ by the construction of $W$, and thus

$$
|x_i - s| \leq \sqrt{N} \sqrt{|e|^2 + |\hat{\xi}|^2} = \sqrt{N} \sqrt{2V}.
$$

Therefore, for any $i \in N$,

$$
\limsup_{t \to \infty} |x_i(t) - s(t)| \leq M(B) \sqrt{N} \sqrt{r(1/k_1)}
$$

if $k_1 = k \lambda_2 - L \geq L^2/p$. From this, the class-$K$ function $\sigma^*$ in (12) and the constant $K$ (11) are found.
B. Discussions

The proof of Theorem 1 enlightens the following:

1) The quantity $|R^T f(t,1_N s)|$ has the meaning of ‘measure of heterogeneity’ in the sense that, if all agents are identical: $f_i(t,s) = f_0(t,s)$ for all $s$ and $i \in \mathcal{N}$, then $R^T f(t,1_N s) = R^T 1_N f_0(t,s) = 0$. More specifically, if we denote the first column of $R^T$ by $r_1$ so that $R^T = [r_1, \hat{R}]$ with a matrix $\hat{R}$, then it follows from $R^T 1_N = 0$ that $r_1 = -\hat{R} 1_{N-1}$. Hence,

$$R^T f(t,s) = [r_1, \hat{R}] \begin{bmatrix} f_1(t,s) \\ \vdots \\ f_N(t,s) - f_1(t,s) \end{bmatrix} = \hat{R} \begin{bmatrix} f_2(t,s) - f_1(t,s) \\ \vdots \\ f_N(t,s) - f_1(t,s) \end{bmatrix}.$$

2) If $|R^T f(t,1_N s)| = 0$, the consensus is made for any positive $k$. This can be easily seen from (16b), where $\lim_{t \to \infty} \tilde{\xi}(t) = 0$ implies $\lim_{t \to \infty} |x_i(t) - x_1(t)| = 0$ for all $i = 2, \ldots, N$, because

$$\tilde{\xi} = R^T x = \hat{R} \begin{bmatrix} x_2 - x_1 \\ \vdots \\ x_N - x_1 \end{bmatrix}.$$

Even in this case, it is seen from (16a) that convergence of $e(t)$ to zero (or, $x_i(t)$ to $s(t)$) requires a certain stability, such as incremental stability, of the averaged dynamics (3).

3) If we bypass (20), the inequality (10) becomes

$$\lim_{t \to \infty} \sup_{t \to \infty} |x_i(t) - s(t)| \leq \lim_{t \to \infty} \sup_{t \to \infty} |R^T f(t,1_N s(t))| \sqrt{\frac{\sqrt{N}}{1/(k\lambda_2 - L)}}.$$

C. Simulation

Theorem 1 means that as the coupling strength $k$ gets larger, the group behavior becomes closer to the trajectory of the averaged dynamics. To see this, let us consider the following group of heterogeneous subsystems:

$$\begin{align*}
    f_1(t,x_1) &= -x_1 + 2.7 \sin(0.35t + 57.3), \\
    f_2(t,x_2) &= -0.75x_2 + 1.4 \sin(0.22t + 40.1), \\
    f_3(t,x_3) &= -0.5x_3 + 4.7 \sin(0.35t + 10.3), \\
    f_4(t,x_4) &= 0.5x_4 + 4.5 \sin(0.13t + 17.2), \\
    f_5(t,x_5) &= 0.75x_5 + 2 \sin(0.01t + 23).
\end{align*}$$

In this case, the averaged dynamics of the subsystems can be obtained as

$$\dot{s} = -s + \frac{1}{5} \{2.7 \sin(0.35t + 57.3) + 1.4 \sin(0.22t + 40.1) + 4.7 \sin(0.35t + 10.3) + 4.5 \sin(0.13t + 17.2) + 2 \sin(0.01t + 23)\}.$$

Although the group includes two unstable agents, the averaged dynamics is not unstable.

Here, we assume the network topology is a ring network in which each node connects to exactly two other nodes, forming a single continuous pathway for signals through each node.\(^3\) The simulation results show that by increasing the coupling strength $k$, the behaviors of all agents approach the trajectory of the averaged dynamics; see Fig. 1. Compared with Fig. 1(a), the plot in Fig. 1(b) shows more collective and synchronized behaviors of the subsystems.

III. Robustness by a Large Number of Agents

In this section, the simulation studies confirm that as the number of agents increases in the network, the averaged dynamics with perturbations and noise approaches the nominal averaged dynamics. Furthermore, we will also illustrate that strong coupling and a large number of agents lead to the synchronization with the nominal averaged dynamics.

A. Noise attenuation effect of a large number of agents

As mentioned in Introduction, we can easily see the effect of a large number of agents, employing the averaged dynamics.

Let us consider the following group of subsystems:

$$f_i(t,x_i) = \frac{1}{2} (-1 + \Delta_i) x_i + 10m_i \sin(2w_i t + \theta_i),$$

where $\Delta_i$, $m_i$, $w_i$, and $\theta_i$ are supposed to be perturbations to the subsystems. In the simulation, we choose them randomly, all of which follow the standard normal distribution $N(0,1)$. From now on, the trajectory $s(t)$ is regarded as the averaged dynamics with perturbations. The nominal averaged dynamics is the trajectory $s_0(t)$, which is the averaged dynamics without perturbations. The nominal averaged dynamics is obtained as $\dot{s}_0 = \frac{1}{2} s_0$ in this

\(^3\)In fact, all the network topologies are assumed to be ring network in simulation examples of this paper.
Fig. 2. The black solid curve and dashed curve represent the trajectories of the averaged dynamics $s(t)$ and the nominal averaged dynamics $s_0(t)$, respectively.

Fig. 3. The black solid curve and dashed curve represent the trajectories of the averaged dynamics $s(t)$ and the nominal averaged dynamics $s_0(t)$, respectively.

Fig. 4. The trajectories (blue solid) of $N$-agent systems with the coupling strength $k$, and the trajectory $s_0(t)$ (black dashed) of the nominal averaged dynamics.

**B. Robustness by a large number of agents with strong coupling**

In the previous subsection, we have illustrated the noise attenuation effect of a large number of agents. From the simulation results, it has been confirmed that the behavior of the averaged dynamics with perturbations and noise can get very closer to that of the nominal averaged dynamics in a large number of agents. Besides, Theorem 1 means that as the coupling strength $k$ increases, the individual behavior becomes closer to the trajectory of the averaged dynamics. Hence, from these two findings, we can expect that the interconnected systems are robust against noise if the number of agents and the coupling strength are sufficiently large.

Let us revisit the first simulation in the subsection III.A. The trajectories of individual systems and the nominal averaged dynamics are illustrated in Fig. 4. Compared with Fig. 4(a) and Fig. 4(b) ($N = 5$), it is considered to have a large number of agents ($N = 50$) in Fig. 4(c) and Fig. 4(d). When $N = 5$ and $k = 5$, it is hard to see that the individual systems achieve synchronization behaviors and get closer to the trajectory of the nominal averaged dynamics (Fig. 4(a)). Even though the coupling gain is sufficiently large ($k = 500$), the group behavior cannot approach the trajectory of the nominal averaged dynamics (Fig. 4(b)). In contrast, when $N = 50$ and $k = 5$, the trajectories of the individual systems are evenly distributed around the trajectory of the nominal averaged dynamics (Fig. 4(c)); see also Fig. 2(b). Moreover, as the coupling strength gets larger ($k = 500$), the group behavior becomes closer to the trajectory of the nominal averaged dynamics (Fig. 4(d)).

Similarly, the second simulation results show that strong coupling and a large number of agents both enhance robustness of the networked group behavior in Fig. 5. We can see that even if some of the nominal subsystems are unstable,
the collective behaviors are robust against the noise as long as both strong coupling and a large number of agents are given.

IV. CONCLUDING REMARKS

In this paper, we have claimed that strong coupling and a large number of agents both enhance robustness of the networked group behavior. While strong coupling guarantees that each agent’s dynamics synchronizes with the averaged dynamics, a large number of agents attenuate the effects of noise and perturbations so that the averaged dynamics approaches close to its nominal one. Consequently, strong coupling and a large number of agents yield that each agent’s dynamics synchronizes with the nominal averaged dynamics. However, contrary to the case of the robustness by strong coupling, this paper has dealt with that of a large number of agents by simulation instead of by theoretical analysis. Thus, our future work will focus on providing a mathematical proof of the robustness by a large number of agents.

REFERENCES