Supplementary Appendix of
“Passivity Framework for Nonlinear State Observer”

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This is a supplementary material for the paper “Passivity Framework for Nonlinear State Observer” written by the same authors. This material contains the examples and proofs which are omitted in the above paper due to the page limit of American Control Conference 2000.

Illustrative Examples

Example 1. Consider a simple system
\[ \begin{align*}
\dot{x}_1 &= x_2 + x_2u^2 \\
\dot{x}_2 &= u
\end{align*} \] (101)
which is uniformly observable in the sense\(^1\) that \(x_1 = y\) and \(x_2 = \dot{y}/(1 + u^2)\).

With this example, Teel [3] pointed out that, for input non-affine systems, the transformability to the triangular form (24) is not necessary for uniform observability, although for affine systems, the transformability to (22) is necessary and sufficient for the uniform observability [4]. Indeed, the input non-affine system (101) cannot be put into the form of (24), but is uniformly observable. Since (101) is uniformly observable, the high-gain observer of the type in [1, 5] exists but requires the knowledge of \(u\). Even though [2] gives an answer without \(\dot{u}\) in this case, design of PSO is another easier answer.

Suppose an observer for (101) as
\[ \begin{align*}
\dot{z}_1 &= z_2 + z_2u^2 + lv \\
\dot{z}_2 &= u + lv
\end{align*} \]
with which the error dynamics is obtained as
\[ \begin{align*}
\dot{e}_1 &= e_2 + e_2u^2 + lv \\
\dot{e}_2 &= lv + y_a - e_1.
\end{align*} \]
Therefore, the zero dynamics becomes
\[ \dot{e}_2 = -\frac{l_2}{l_1}(1 + u^2)e_2 \]
for which the condition C1 holds with \(V(e_2) = \frac{1}{2}e_2^2\), \(l_1 = 1\) and \(l_2 = 1\). Moreover, C2 holds with \(\phi_1 = \phi_2 = 1 + u^2\) in (21). Indeed, (21) becomes
\[ |e_2 - (1 + u^2)e_1| + e_1(1 + u^2)(e_2 + e_1) \leq (1 + u^2)|e_2|^2. \]

Thus, Theorem 1 gives a PSO for (101).

Example 2. Consider a system
\[ \begin{align*}
\dot{x}_0 &= -(1 + x_0^2)x_0 + (x_1 - 1)u \\
\dot{x}_1 &= -x_1u^2 \\
\dot{x}_2 &= -x_2^3 - 2x_1(1 + u^2) \\
y &= x_2,
\end{align*} \] (102)
to which the several approaches in Section 4 are not applicable.

For (102), the vector fields of error dynamics are obtained as
\[ \begin{align*}
F_1(e; x, u) &= \left[ -(e_0 + x_0)(1 + (e_0 + x_0)^2) + x_0(1 + x_0^2) + e_1u \right] \\
F_2(e; x, u) &= -(e_2 + x_2)^3 + x_2^2 - 2e_1(1 + u^2).
\end{align*} \]

Let \(L_1 = [l_0, l_1]^T\) and \(L_2 = 1\), then the left-hand term of (20) becomes
\[ e_0[-(e_0 + x_0)(1 + (e_0 + x_0)^2) + x_0(1 + x_0^2) + l_0(2e_1(1 + u^2))]\]
\[ + e_0e_0u + e_1[-e_1u^2 + l_1(2e_1(1 + u^2))] \]
with \(V(e_0, e_1) = \frac{1}{2}(e_0^2 + e_1^2)\). Here, since
\[ e_0[-(e_0 + x_0)(1 + (e_0 + x_0)^2) + x_0(1 + x_0^2)]\]
\[ = -(e_0^2 + 3e_0e_1 + (3x_0^2 + 1)) \leq -e_0^2, \]
choose \(l_0 = 0\). Similarly, by taking \(l_1 = -1\) the last term becomes \(-2 + 3u^2e_1^2\). Therefore, (20) becomes that
\[ D_{(e_0, e_1)}V \left[ F_1 - L_1 L_2^{-1} F_2 \right] \leq -e_0^2 + e_0 e_1 u - 2e_1^2 - 3u^2 e_1^2 \]
\[ = -\frac{1}{2}e_0^2 - 2e_1^2 - 5\frac{u^2}{2}e_1^2 - \frac{1}{2}(e_0 - e_1)^2 \]
\[ \leq -\frac{1}{2}(e_0^2 + e_1^2) \]
which shows that the condition C1 holds.

For the condition C2, (21) is written as
\[ |e_0 e_1| \left\{ \begin{array}{ll}
\frac{-e_2 u}{e_2 u^2} & + 0 \\
\frac{1}{2} & \frac{-e_2^3}{e_2^2} \left[ -(e_2 + x_2)^3 + x_2^2 + 2e_2(1 + u^2) \right]
\end{array} \right. \]
\[ + e_2 \left[ -(e_2 + x_2)^3 + x_2^2 - 2(e_1 - e_2)(1 + u^2) \right] \]
\[ = \left[ -e_0 e_2 u + e_1 \left\{ e_2 u^2 - (e_2 + x_2)^3 + x_2^2 \right\} \right. \]
\[ + e_2 \left\{ -(e_2 + x_2)^3 + x_2^2 + 2e_2(1 + u^2) \right\} \]
\[ \leq |u| |e_0| |e_2| + |e_2^2 + 3x_2 e_2 + 3x_2^2 - u^2| e_1 |e_2| \]
\[ + |e_2^2 + 3x_2 e_2 + 3x_2^2 - 2 - 2u^2| |e_2|^2. \]
Thus, C2 is satisfied with $\phi_1(u, y, y_a) = |e_2^2 + 3x_2e_2 + 3x_2^2 - 2 - 2u^2|$ and $\phi_2(u, y, y_a) = 2|u| + |e_2 + 3x_2e_2 + 3x_2^2 - u^2|$. Then, Theorem 1 gives a PSO for (102).

**Appendix: Proof of Lemma 2**

**Proof:** In fact, the claim follows from [6, Lemma 2] or [7, Appendix] with a simple trick. Let $\mu(u, y, y_a) := k(u, y, y_a)y_a$. Then, by [6, Lemma 2], it follows that there are a continuous function $\kappa$ and a $K$ function $\rho$ such that

$$
\|\mu(u + d_1, y + d_2, y_a + d_3) - \mu(u, y, y_a)\| \leq \kappa(u + d_1, y + d_2, y_a + d_3)\rho\left(\|\alpha(d_1, d_2, d_3)\|\right)
$$

for all $u$, $y$, $y_a$, $d_1$, $d_2$ and $d_3$. Taking $d_1 = 0$, $d_2 = d$, $d_3 = -d$ and $\rho(s) = \rho(\sqrt{2}s)$ proves the claim.

**Appendix: Proof of Lemma 3**

**Proof:** The proof is similar to [8]. With the modified observer (15),

$$
\dot{V}(x, e) = D_xV \cdot f(x, u) + D_yV \cdot (F(e; c, u) - l(e + x, u)k(u, y, d, y_a - d)y_a - d) - k(u, y, y_a)y_a
$$

By (13) and (14),

$$
\dot{V}(x, e) \leq -\alpha_3(||e||) + ||y_a|| \cdot \|k(u, y, y_a - d) - d\| - k(u, y, y_a)y_a
$$

$$
= -\alpha_3(||e||)
$$

subject to $||u|| \leq s_1$, $||y|| \leq s_2$, $||d|| \leq s_3$ and $||y_a|| \leq \rho(s_3) + s_3$. Then,

$$
||e|| \geq \alpha_3^{-1}(2\tilde{\gamma}(||u||, ||y||, ||d||)) \Rightarrow \dot{V} \leq -\frac{1}{2}\alpha_3(||e||).
$$

Therefore, (16) follows with $
\gamma(s_1, s_2, s_3) = \alpha_3^{-1}(\alpha_2(\alpha_3^{-1}(2\tilde{\gamma}(s_1, s_2, s_3))))$ [9].

**Appendix: Proof of Theorem 1**

**Proof:** By change of coordinates $\xi_1 = e_1 - L_1L_2^{-1}e_2$ and $\xi_2 = e_2$, the augmented error dynamics (18) becomes

$$
\dot{\xi}_1 = F_1(\xi_1 + L_1L_2^{-1}\xi_2, \xi_2; x_1, x_2, u)
$$

$$
- L_1L_2^{-1}F_2(\xi_1 + L_1L_2^{-1}\xi_2, \xi_2; x_1, x_2, u)
$$

(103)

$$
\dot{\xi}_2 = F_2(\xi_1 + L_1L_2^{-1}\xi_2, \xi_2; x_1, x_2, u) + L_2v
$$

(104)

In this coordinates, it is clear that the zero dynamics is

$$
\dot{\xi}_1 = F_1(\xi_1, 0; x_1, x_2, u) - L_1L_2^{-1}F_2(\xi_1, 0; x_1, x_2, u)
$$

which are the same representation as (19). Using the abbreviation $f_1^*$:

$$
\dot{\xi}_1 = f_1^*(\xi_1, x, u) + F_1(L_1L_2^{-1}\xi_2, \xi_1 + x_1, x_2, u)
$$

$$
- L_1L_2^{-1}F_2(L_1L_2^{-1}\xi_2, \xi_1 + x_1, x_2, u)
$$

(105)

where the term $F_1 - L_1L_2^{-1}F_2$ vanishes when $\xi_2 = 0$. Now, let a storage function be

$$
W(x, \xi) := W(x, \xi_1) + \frac{1}{2}\xi_2^2L_2^{-1}\xi_2
$$

which clearly satisfies

$$
\alpha_1(||\xi||) \leq W(x, \xi) \leq \alpha_2(||\xi||)
$$

where $\alpha_1$ and $\alpha_2$ are $K_\infty$ functions as in Definition 1. By Conditions C1 and C2, the time derivative of $W$ along the trajectory of (103) satisfies

$$
W = D_xW f(x, u) + D_yW f_1^*(\xi_1, x, u)
$$

$$
+ D_\xi_1W (F_1 - L_1L_2^{-1}F_2)(L_1L_2^{-1}\xi_2, \xi_2; x_1, x_2, u)
$$

$$
+ \xi_2^2L_2^{-1}F_2(\xi_1 + L_1L_2^{-1}\xi_2, \xi_2; x_1, x_2, u) + \xi_2^2v
$$

$$
\leq -\psi_1(||\xi||) + \phi_1(u, x_2, \xi_2)||\xi_2||^2
$$

$$
+ \phi_2(u, x_2, \xi_2)\psi_1^2(||\xi||)||\xi_2|| + \xi_2^2v
$$

Therefore, by applying $v = -k(u, y, y_a)y_a + \bar{v}$,

$$
W \leq \frac{3}{4}\psi_3(||\xi||) - \xi_2||\xi_2||^2
$$

$$
+ (\phi_1 - \phi_2)||\xi_2||^2 - (\frac{1}{2}\psi_3^2 - \phi_2||\xi_2||)^2 + \xi_2^2\bar{v}.
$$

This leads to the PSUP with respect to $(\xi, u)$, that is with respect to $(e, u)$.

**Appendix: Proof of Lemma 4**

**Proof:** Let $z_i = \theta^{i-1}x_i$ for $1 \leq i \leq n$. Then, in $z$-coordinates, (26) becomes

$$
\dot{z}_1 = \theta z_2 - l_1z_1 + \gamma_1(z_1, d)
$$

$$
\dot{z}_2 = \theta z_3 - \frac{1}{\theta}l_2z_1 + \frac{1}{\theta}z_2(z_1, \theta z_2, d)
$$

$$
\vdots
$$

$$
\dot{z}_n = -\frac{1}{\theta^{n-1}}l_nz_1 + \frac{1}{\theta^{n-1}}\gamma_n(z_1, \cdots, \theta^{n-1}z_n, d).
$$

p. 2
By the Lipschitz property and the triangular structure, there exists a constant \( \rho_1 \), which is independent of \( \theta \), such that
\[
\frac{1}{\theta^{1-i}} h(\eta_1, \cdots, \theta^{i-1} \eta_i, d) \leq \rho_1 \|z\|, \quad 1 \leq i \leq n.
\]
Therefore, with \( l_i = \theta^i a_i \), the system can be written as
\[
\dot{z} = \theta A_i z + \tilde{\gamma}(z, d, \theta)
\]
where \( \tilde{\gamma} \) satisfies \( \|\tilde{\gamma}(z, d, \theta)\| \leq \rho\|z\| \) with \( \rho = n \cdot \max \rho_i \).

Finally, the derivative of \( V(z) = \frac{1}{2} z^T P_0 z \) along the trajectory is
\[
\dot{V} \leq -\theta \|z\|^2 + \rho \|P_0\| \|z\|^2
\]
which is negative definite for \( \theta > \rho \|P_0\| \).

Appendix: Condition T3 implies C2 (Section 4.2)

For simplicity, the proof is performed in the transformed coordinates. In other words, without loss of generality, we suppose that the system
\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2, u) \\
\dot{x}_2 &= f_2(x_1, x_2, u)
\end{align*}
\]
(106)
satisfies the conditions T1 and T3 with a positive definite matrix
\[
P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}.
\]
Indeed, if the given system (27) satisfies T1 and T3 with
\[
P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}
\]
then, by the linear change of coordinates
\[
T := \begin{bmatrix} I & P_{11} \quad P_{12} \\ 0 & P_{21} \quad P_{22} \end{bmatrix},
\]
and by redefining the transformed system as (106), it can be shown that (106) satisfies T1 and T3 with \( P_1 = P_{11} \) and \( P_2 = P_{22} = P_1 \) and \( P_{21} = P_{11}^{-1} P_{22} \). In addition, since (106) satisfies T1 with (107), it can be shown that C1 still holds in this coordinates with \( V(e_1) = \frac{1}{2} e_1^T P_1 e_1, L_1 = 0 \) and \( L_2 = P_2^{-1} \), by repeating the argument of Section 4.2. Note that in this case, \( \psi_3(\|e_1\|) \) of C1 is equal to \( k_1 \|e_1\|_2^2 \) with \( k_1 \) in T1.

Now, suppose T3 holds for (106) and (107), i.e., there is a continuous function \( p(u) \) such that
\[
|\begin{bmatrix} x_1^T P_1 D_1 f_1(q, u) x_1 + x_2^T P_2 D_1 f_2(q, u) x_1 + x_1^T P_1 D_2 f_1(q, u) x_2 + x_2^T P_2 D_2 f_2(q, u) x_2 \end{bmatrix}^2 
\leq \rho u \|x_1\|^2 + \rho u \|x_2\|^2
\]
for all \( q, x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \). Then, it follows by Lemma A.1 that
\[
|\begin{bmatrix} x_1^T P_1 D_1 f_1(q, u) x_1 + x_2^T P_1 D_2 f_1(q, u) x_2 + x_1^T P_2 D_1 f_2(q, u) x_2 + x_2^T P_2 D_2 f_2(q, u) x_2 \end{bmatrix}^2 
\leq \rho u \|x_2\|^2 + \rho u \|x_2\| \|x_1\| \|x_2\|\]
(108)
for all \( q, x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \).\footnote{Actually, Remark 2 [10] shows that the Conditions T1-T3 are invariant to the linear change of coordinates.}

**Lemma A.1** Let \( x \in \mathbb{R}^n, y \in \mathbb{R}^m, z \in \mathbb{R}^p \) and \( u \in \mathbb{R}^p \), and let \( L, M \) and \( N \) be matrix-valued functions with appropriate dimensions. If there exists a continuous function \( c(u) \) such that
\[
\begin{align*}
|\begin{bmatrix} x^T L(z, u)x + x^T M(z, u)y + y^T N(z, u)y \end{bmatrix}^2 
\leq c(u) \|y\|^2 + c(u) \|x\|^2
\end{align*}
\]
(109)
for all \( x, y, z \) and \( u \), then there exists a continuous function \( d(u) \) such that
\[
|\begin{bmatrix} x^T M(z, u)y + y^T N(z, u)y \end{bmatrix}^2 
\leq c(u) \|y\|^2 + c(u) \|x\| \|y\|
\]
(110)
for all \( x, y, z \) and \( u \).

**Proof:** First, by letting \( x = 0 \) in (109), it holds that
\[
|\begin{bmatrix} y^T N(z, u)y \end{bmatrix}^2 \leq c(u) \|y\|^2.
\]
Moreover, it is easy to see that the diagonal terms of \( N(z, u) \) is bounded with respect to \( z \) by fixing \( y = \delta_i \), where \( \delta_i \) is such that its \( i \)-th element is 1 and others are 0. Now, we show element-wise that \( M(z, u) \) is bounded with respect to \( z \). That is, \( (i, j) \)-th element of \( M \), say \( M_{ij}(z, u) \), is bounded since, when \( x = \delta_i, y = \delta_j \), the equation (109) becomes
\[
|\begin{bmatrix} M_{ij}(z, u) + N_{ij}(z, u) \end{bmatrix}^2 \leq 2c(u)
\]
where \( N_{ij} \) is bounded. Using their boundedness, it is clear that (110) holds.

Now, consider the condition C2. Since \( V(e_1) = \frac{1}{2} e_1^T P_1 e_1, L_1 = 0 \) and \( L_2 = P_2^{-1} \), C2 is simplified as
\[
|\begin{bmatrix} e_1^T P_1 F_1(0, e_2, e_1 + x_1, x_2, u) + e_1^T P_2 F_2(e_1, e_2, x_1, x_2, u) \end{bmatrix}^2 
\leq \phi_1(u, x, e_2)^2 + \phi_2(u, x, e_2)^2 \sqrt{k_1 \|e_1\| \|e_2\|}.
\]
(111)
Then, for each \((e, x, u), \) there are \( q \in \mathbb{R}^n \) such that
\[
\begin{align*}
e_1^T P_1 F_1(0, e_2, e_1 + x_1, x_2, u) + e_1^T P_2 F_2(e_1, e_2, x_1, x_2, u) \\
= e_1^T P_1 D_2 f_1(q, u) e_2 + e_1^T P_2 D_2 f_2(q, u) e_1 \\
+ e_2^T P_2 D_2 f_2(q, u) e_2
\end{align*}
\]
by the following Lemma A.2.

**Lemma A.2** For each \( e, x \) and \( u \), there exists \( q \in \mathbb{R}^n \) such that
\[
\begin{align*}
e_1^T P_1 (f_1(e_1 + x_1, e_2 + x_2, u) - f_1(e_1, x_2, u)) \\
+ e_2^T P_2 (f_2(e_1 + x_1, e_2 + x_2, u) - f_2(x_1, x_2, u)) \\
= e_1^T P_1 D_2 f_1(q, u) e_2 + e_1^T P_2 D_2 f_2(q, u) e_1 \\
+ e_2^T P_2 D_2 f_2(q, u) e_2
\end{align*}
\]

**Proof:** Let
\[
f^*(x, z, u, p) = \begin{bmatrix} p_1^T P_1 & p_2^T P_2 \end{bmatrix} \begin{bmatrix} f_1(z_1, z_2, u) - f_1(z_1, z_2 - x_2, u) \\
f_2(z_1, z_2, u) - f_2(z_1 - x_1, z_2 - x_2, u) \end{bmatrix}
\]
where \((z, u, p)\) is supposed to be parameters. Then, since \( f^*(0, z, u, p) = 0 \) and \( f^* : \mathbb{R}^n \rightarrow \mathbb{R} \) for each \((z, u, p)\), by the
mean-value theorem [11, p.59], for each $x$, $z$, $u$ and $p$, there exists $q$ such that

$$f^*(x, z, u, p) = D_x f^*(q, z, u, p)x$$

$$= [p_1^T P_1 \quad p_2^T P_2] \begin{bmatrix} 0 & D_2 f_1(q, u) \\ D_1 f_2(q, u) & D_2 f_2(q, u) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. $$

Finally, substituting $x = e$, $p = e$ and $z = e + x$ proves the claim. ■

Remark A.1 Lemma A.2 is not a trivial consequence of mean-value theorem. When $f : \mathbb{R}^n \to \mathbb{R}^n$ ($n > 1$), it is not true in general that, for each $x$, there is a $q$ such that

$$f(x) = Df(q)x.$$ 

This can be seen by a counterexample that $f(x_1, x_2) = [x_1^T, \exp(x_1) - 1]^T$ when $x_1 = x_2 = 1$.

Finally, it is easy to see that (108) implies (111) which claims that T3 implies C2, with $\phi_1 = p(u)$ and $\phi_2 = r(u)/\sqrt{R_1}$.

Appendix: Proof of Theorem 4

Proof: The conditions B1–B2 can be easily shown to be invariant under the linear change of coordinates. Therefore, we assume, without any loss of generality, the given system (32) has the output $y = Cx = x_2$ where $C = [0, I]$ and the system satisfies B1 and B2 with the block diagonal matrix $P > 0$, i.e.

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}.$$ 

This can be verified by two step transformations of coordinates ($T = T_2 T_1$). In fact, after transforming into the intermediate coordinates with $T_1 = [D^T C^T]^T$, apply the second transformation $T_2 = \begin{bmatrix} I & P_{11}^{-1} P_{12} \\ 0 & I \end{bmatrix}$, where $P_{ij}$ is the $(i,j)$-th block element of $P$ which satisfies B1 for the intermediate coordinates.

Then the system (32) can be written as

$$\dot{x}_1 = A_{11} x_1 + A_{12} x_2,$$

$$\dot{x}_2 = A_{21} x_1 + A_{22} x_2 + P_2^{-1} h(x_1, x_2, u).$$

On the other hand, note that $Q > 0$ implies $Q_{11} > 0$, by which and B1, it follows that $P_1 A_{11} + A_{11}^T P_1 < 0$. Now it can be easily checked that this system satisfies C1 with $V(e_1) = \frac{1}{2} e_1^T P_1 e_1$, $L_1 = 0$ and $L_2 = P_2^{-1}$. The condition C2 also follows from B2. Therefore, Theorem 1 provides the state observer (PSO) for (32). ■

Appendix: Proof of Theorem 5

Proof: By $\xi = x_1 - L^* x_2$ and $x_2 = x_2$, the given system (17) is transformed into

$$\dot{\xi} = f_1(\xi + L^* x_2, x_2, u) - L^* f_2(\xi + L^* x_2, x_2, u) $$ (112a)

$$\dot{x}_2 = f_2(\xi + L^* x_2, x_2, u) $$ (112b)

$$y = x_2. $$ (112c)

Since $x_2$ is measurable, we discard $x_2$-dynamics and construct an observer for (112a) as in (5). Then, the augmented error dynamics becomes ($e = z - \xi = z - (x_1 - L^* x_2)$)

$$\dot{\xi} = f(\xi, u) $$

$$\dot{e} = f_1(e, 0; x_1, x_2, u) - L^* f_2(e, 0; x_1, x_2, u).$$

Finally, it can be shown that $z(t) \to \xi(t) = x_1(t) - L^* x_2(t)$ as $t \to \infty$ by utilizing Lemma 1 with C1.

References


