Passification of SISO LTI Systems Through a Stable Feedforward Compensator

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Abstract: The paper addresses the design problem of a parallel feedforward compensator for a class of SISO LTI systems, that renders the augmented system to be of minimum phase and have relative degree one. The plant can be of non-minimum phase and/or have high relative degree. Under these setups, we derive an existence condition of a stable feedforward compensator and propose a design algorithm based on the simultaneous stabilization of two systems. An illustrative example is also given.

Keywords: Parallel feedforward compensator, non-minimum phase systems, passivation, output feedback.

1. INTRODUCTION

The passivity-based control has been widely studied for the stabilization problems and many other applications \cite{2}, \cite{7}, \cite{8}. However, the condition that the plant is weakly minimum phase and has relative degree one is not only sufficient but also necessary for the plant to be feedback passive. Thus, the class of systems to which the passivity-based approach is applicable is restricted. If the system is of non-minimum phase and/or has relative degree greater than one, one cannot passivate the system by feeding back the state. A choice to overcome this restriction is to use a parallel feedforward compensator (PFC) shown in Fig. 1. We consider the problem of designing a PFC which renders the augmented plant to be of minimum phase and to have relative degree one for such SISO LTI systems.

For LTI systems, the idea of passivation using a PFC has been researched by several authors \cite{1}, \cite{5}, \cite{9}. In \cite{1} and \cite{5}, it is applied to a simple adaptive control scheme (SAC). In spite of its attractive performance, it has not received much attention since the plant has to satisfy the almost strictly positive real (ASPR) condition. If a LTI system is of minimum phase and has relative degree one, then the system is rendered strictly passive (or strictly positive real (SPR)) by a static output feedback (SOF) law. In that case, the system is called almost strictly passive (ASP or equivalently, ASPR). The authors of \cite{1} showed that if a plant is stabilized by a compensator $H^{-1}(s)$, then the non-ASPR plant can be made ASPR approximately by augmenting a PFC $H(s)$ to the plant. In \cite{5}, a concrete PFC design approach has been proposed in the frequency domain. Unlike the result of \cite{1}, this method does not require a prior knowledge of a stabilizing controller, but the plant has to satisfy some assumptions such as minimum phaseness.

The work \cite{9}, which mainly motivates our work, provided the first result of designing a PFC in the state-space. It even covered non-square systems, i.e., systems having different numbers of inputs and outputs. It also provided a method for designing a PFC and a squaring down matrix where the dimension of the PFC is same as that of the control input. Moreover, by \cite{9}, the zero dynamics of the augmented system with the PFC was turned out to be equivalent to a closed-loop system of a virtual system with a static output feedback (SOF) law, and a necessary and sufficient condition for the existence of such SOF gain was given. However, the existence condition for the SOF gain is not easy to check since the condition is given by a Riccati-like inequality.

The first purpose of this paper is to provide a method to design a PFC even when the SOF problem \cite{9} does not have a solution, at the expense of a higher-order PFC. It is shown that the zero dynamics of the parallel interconnection of a SISO LTI plant and a PFC is equivalent to a closed-loop system of a virtual system with a dynamic output feedback (DOF) law rather than a SOF law. Moreover, unlike all the above results, we impose one more condition to the design problem; namely, the PFC itself should be stable. This condition is necessary from the practical point of view. In many cases, control engineers are reluctant to use an unstable compensator be-
cause of system integrity. For example, if a sensor or actuator fails and the feedback loop opens, then the unstable PFC blows up which is undesirable. With this additional condition, it is shown that our problem is closely related to the simultaneous stabilization problem of two systems. From this observation, we provide an existence condition for the stable PFC and its design algorithm.

The paper is organized as follows. In Section 2, the design problem of a stable PFC and the state space characterization of the problem are presented. Based on the simultaneous stabilization problem, Section 3 provides an existence condition for the PFC in the frequency domain and gives a design algorithm. In Section 4, an illustrative example for the problem is given and finally, Section 5 concludes the paper.

**Notation:** For the constructive development, we briefly introduce some notation. The set of all complex numbers having nonnegative real parts is denoted as $\mathbb{C}_{\geq 0}$, i.e., $\mathbb{C}_{\geq 0} := \{ s \in \mathbb{C} : \text{Re}(s) \geq 0 \}$. The sets $\mathbb{C}_{< 0}, \mathbb{C}_{> 0}$, and $\mathbb{R}_{\geq 0}$ are defined analogously. On the other hand, $\mathbb{R}_{\geq 0}^\infty := \mathbb{R}_{\geq 0} \cup \{ s = +\infty \}$. The symbol $[a; b]$ stands for the stack of two vectors (or matrices) $a$ and $b$.

### 2. DESIGN PROBLEM OF PFC

In this section, we formulate the problem of designing a stable PFC which makes the parallel interconnected system be of minimum phase and have relative degree one, and give the state space characterization of the problem.

We begin by considering a SISO plant $P$ given as

$$
P : \left\{ \begin{array}{l}
\dot{x} = A_p x + B_p u, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}, \\
y_p = C_p x, \quad y \in \mathbb{R},
\end{array} \right.
$$

(1)

in which the triplet $(A_p, B_p, C_p)$ is controllable and observable. For the plant $P$ having relative degree greater than 1, the PFC has to have relative degree 1 so that the parallel interconnection has relative degree 1. Thus, we let a PFC $V$ be of the form

$$
V : \left\{ \begin{array}{l}
\dot{\eta} = A_0 \eta + A_{10} y_v, \quad \eta \in \mathbb{R}^n, \\
y_v = A_{10} \eta + a_{11} y_v + b_v u, \quad y \in \mathbb{R},
\end{array} \right.
$$

(2)

where $y_v$ is the output of the PFC. Then the parallel connection $P + V$ as shown in Fig. 1 is written as

$$
P + V : \left\{ \begin{array}{l}
\dot{z} = \begin{bmatrix} A_p & 0 & 0 \\ 0 & A_{10} & A_{11} \\ 0 & a_{11} & b_v \\
\end{bmatrix} z + \begin{bmatrix} B_p \\ 0 \\ b_v \\
\end{bmatrix} u, \\
y = \begin{bmatrix} C_p & 0 & 1 \end{bmatrix} z, \\
\end{array} \right.
$$

(3)

where $z = [x; \eta; y_v]$.

In order to solve the problem, the quantities $A_{00}, A_{01}, A_{10}, a_{11},$ and $b_v$ have to be determined so that the parallel interconnected system (3) has relative degree 1 and is of minimum phase, while the PFC (2) is stable. Note that the time derivative of the output $y$ is given by

$$
y = C_p A_p x + A_{10} \eta + a_{11} y_v + (C_p B_p + b_v) u
$$

(4)

and if $b_v \neq 0$ and $C_p B_p + b_v \neq 0$, then the interconnected system (3) has relative degree 1. Form this observation, we give the following.

**Theorem 1:** The design problem of a stable PFC $V$ is solved if there exist $k, A_{00}, A_{01}, A_{10},$ and $a_{11}$ such that the following conditions hold simultaneously.

(a) $k \neq 0$ and $k \neq 1/(C_p B_p)$.

(b) The matrices $A_v$ and $A_0$ are Hurwitz, in which

$$
A_0 := \begin{bmatrix} A_p - k B_p C_p A_p + k B_p a_{11} C_p - k B_p A_{10} & -A_{10} & a_{11} \end{bmatrix},
$$

$$
A_v := \begin{bmatrix} A_{00} & A_{01} \end{bmatrix} A_{10} a_{11}.
$$

In this case, $b_v$ is determined as $b_v = -C_p B_p + (1/k)$.

**Proof:** Since the matrix $A_v$ is the system matrix of (2), the stability of the PFC $V$ is ensured. Now we show that the Hurwitz matrix $A_0$ implies the minimum phaseness of the interconnected system (3). We compactly rewrite (3) as

$$
\dot{z} = A z + B u,
$$

$$
y = C z.
$$

(5)

Since $1/k = C_p B_p + b_v \neq 0$, the system (5) has the well-defined relative degree 1 and hence its zero dynamics is obtained from $\dot{z} = [A z + B \alpha(z)]|_{z \in \mathbb{R}^n}$, where $\alpha(z)$ and $Z^*$ are defined as follows (see [7]).

$$
\alpha(z) := - C A - C B Z^* = - \begin{bmatrix} C_p A_p A_{10} a_{11} \end{bmatrix} z,
$$

$$
Z^* := \{ z | C z = 0 \} = \{ C_p x + y_v = 0 \}.
$$

Noting that $k = 1/(C_p B_p + b_v)$, we obtain

$$
\dot{\tilde{z}} = [A z + B \alpha(z)]|_{z \in \mathbb{R}^n} = \begin{bmatrix} A_p - k B_p C_p A_p & -k B_p A_{10} \\ 0 & A_{10} - k b_v A_{11} \\
\end{bmatrix} \begin{bmatrix} z \\ y_v = -C_p x. \end{bmatrix}
$$

As a consequence, the zero dynamics of (3) is derived as

$$
\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A_p - k B_p C_p A_p + k B_p a_{11} C_p - k B_p A_{10} & -A_{10} & a_{11} \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix}.
$$

This completes the proof. 

**Remark 1:** In the case of the plant $P$ having relative degree greater than 1, one can show that Theorem 1 is also necessary. Suppose that the problem is solved. Then, this implies that the designed $V$ has relative degree 1 and thus, under a suitable coordinate change, the PFC $V$ is brought into the normal form

$$
\dot{\eta} = A_{00} \eta + A_{01} y_v,
$$

$$
y_v = A_{10} \eta + a_{11} y_v + b_v u,
$$

$$
\begin{bmatrix} x \\ \eta \end{bmatrix} = \begin{bmatrix} A_p - k B_p C_p A_p + k B_p a_{11} C_p - k B_p A_{10} & -A_{10} & a_{11} \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix}.
$$
where \( b_{uv} \neq 0 \). With the normal form, the parallel interconnected system is written as

\[
\dot{z} = \begin{bmatrix} A_p & 0 & 0 \\ 0 & A_{e00} & A_{e01} \\ 0 & A_{e10} & a_{e11} \end{bmatrix} z + \begin{bmatrix} B_p \\ 0 \\ b_{uv} \end{bmatrix} u, \\
y = \begin{bmatrix} C_p & 0 & 1 \end{bmatrix} z.
\]  
(6)

Defining \( k := 1/(C_p B_p + b_{uv}) = 1/b_{uv}, A_{00} := A_{e00}, A_{01} := A_{e01}, A_{10} := A_{e10}, \) and \( a_{11} := a_{e11}, \) one observes that the conditions in Theorem 1 are satisfied if and only if there exists a single DOF controller; i.e.,

\[
A_0 = \begin{bmatrix} A + B a_{11} C & B A_{10} \\ A_{01} C & A_{00} \end{bmatrix}.
\]

In fact, the matrix is the same as the closed-loop system matrix of a plant

\[
\dot{x} = A\dot{x} + B\dot{u}, \quad \ddot{y} = \dot{C}\dot{x},
\]  
(7)

controlled by a dynamic output feedback controller

\[
\dot{\zeta} = A_0\zeta + A_{01}\ddot{y}, \quad \ddot{u} = A_{10}\zeta + a_{11}\ddot{y}.
\]  
(8)

In other words, the matrix \( A_0 \) is Hurwitz if an output feedback controller (8) is designed to stabilize the system (7). An example of such a controller is the observer-based controller; i.e., \( A_{00} = A - BK - LC, A_{01} = L, A_{10} = -K, \) and \( a_{11} = 0 \) where \( A - BK \) and \( A - LC \) are Hurwitz. Note that the gain matrices \( K \) and \( L \) always exist because the triplet \((A, B, C)\) is controllable and observable by Lemma 3 in Appendix.

On the other hand, it is seen that the spectrum of \( A_e \) is the same as that of the matrix \([a_{11} \ A_{10}; A_{01} \ A_{00}]\), which can be regarded as the closed-loop system matrix of a single integrator \( \dot{x} = \ddot{u}, \ddot{y} = \dot{x} \) and its stabilizing controller (8). Therefore, the conditions in Theorem 1 is satisfied if and only if there exists a single DOF controller (8) which stabilizes the system (7) and the single integrator simultaneously.

To tackle the simultaneous stabilization problem, we move on to the frequency domain. Let \( P_1(s) \) and \( P_2(s) \) be the transfer functions of the single integrator and the system (7), respectively. That is,

\[
P_1(s) = \frac{1}{s}, \quad P_2(s) = \dot{C}(sI - A)^{-1} B.
\]  
(9)

Our problem is to design a controller which internally stabilizes \( P_1(s) \) and \( P_2(s) \) simultaneously, where the internal stability implies that the pole/zero cancellation in \( C_+ \) does not occur during the computation of the closed-loop transfer function\(^1\).

To find the existence condition for the simultaneous stabilizing controller for \( P_1(s) \) and \( P_2(s) \), we bring the concept of strong stability\(^2\). There is a known relation between the simultaneous stabilizability and the strong stabilizability.

For given \( P_1(s) \) and \( P_2(s) \), bring in coprime factorizations as

\[
P^* = \frac{N}{M}, \quad N = N_2 M_1 - N_1 M_2, \quad M = N_2 X_1 + M_2 Y_1.
\]

**Lemma 1:** [10] The transfer functions \( P_1(s) \) and \( P_2(s) \) are simultaneously stabilizable if and only if \( P^*(s) \) is strongly stabilizable.

**Lemma 2:** [10] The system \( P^*(s) \) is strongly stabilizable if and only if it has an even number of real poles between every pair of real zeros in \( \text{Re}(s) \geq 0 \). If \( P^* \) has a zero at \( s = +\infty \), this zero must be included among the real zeros of \( P^* \).

The property described in Lemma 2 is called the parity interfacing property [10]. Using these two lemmas, we can find a condition that a simultaneous stabilizing controller for \( P_1(s) \) and \( P_2(s) \) in (9) exists; i.e., a condition guaranteeing the existence of a stable PFC \( V \).

**Theorem 2:** For the given plant \( P(s) = C_p(sI - A_p)^{-1} B_p = n_{p_2}(s)/d_{p_2}(s) \), a stable PFC \( V \) exists if \( n_{p_2}(s) \) has the same signs for all the nonnegative real roots of \( d_{p_2}(s) \).

**Proof:** From the previous discussion, we know that a stable PFC \( V \) exists if \( P_1(s) \) and \( P_2(s) \) are simultaneously stabilizable. Now let us obtain \( P^*(s) \) in Lemma 2 in terms of the given plant \( P(s) \).

First, a simple coprime factorization for \( P_1(s) \) is given as \( N_1 = 1/(s + \mu) \) and \( M_1 = s/(s + \mu) \) for any \( \mu > 0 \) with \( X_1 = 1 \) and \( Y_1 = 1 \). Next, let \( P_2(s) = \dot{C}(sI - A)^{-1} B = n_{p_2}(s)/d_{p_2}(s) \). The polynomials \( n_{p_2}(s) \) and \( d_{p_2}(s) \) are coprime since \((A, B, C)\) is controllable and observable. Then, a coprime factorization for \( P_2(s) \) has the form \( N_2 = n_{p_2}(s)/m(s), M_2 = d_{p_2}(s)/m(s) \) for some stable polynomial \( m(s) \) of the order being the same as that of \( d_{p_2}(s) \) with \( X_2 \) and \( Y_2 \) such that \( N_1 X_2 + M_2 Y_2 = 1 \). Therefore,

\[
P^*(s) = \frac{N_2 M_1 - N_1 M_2}{N_2 X_1 + M_2 Y_1} = \frac{sn_{p_2} - d_{p_2}}{(s + \mu)(\mu n_{p_2} + d_{p_2})}.
\]

To represent \( P_2(s) \) and \( P^*(s) \) in terms of the given original plant \( P(s) \), suppose that without loss of generality, \( A_p, B_p, \) and \( C_p \) are in the controllable canonical

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\(^1\)About the internal stability, see [10, p.35]

\(^2\)A plant is said to be strongly stabilizable if the internal stabilization can be achieved with a controller being itself stable.
To simplify, rewrite
\[ A_p = \begin{bmatrix} -a_1 & \cdots & -a_{n-1} & -a_n \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}, \quad B_p = \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \]
\[ C_p = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}, \]
where \( P(s) = (c_1 s^{n-1} + \cdots + c_n)(s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n) \). Then, \( \bar{A} = A_p - k B_p C_p A_p, \bar{B} = -k B_p, \) and \( \bar{C} = -C_p \) are computed as
\[ \bar{A} = \begin{bmatrix} -a_1 + kc_1 a_1 - kc_2 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -a_{n-1} + kc_1 a_{n-1} - kc_n - a_n + kc_1 a_n \end{bmatrix}, \]
\[ \bar{B} = \begin{bmatrix} -k & 0 & \cdots & 0 \end{bmatrix}^T, \quad \bar{C} = [-c_1 \cdots -c_n]. \quad (10) \]
From (10), \( P_2(s) \) is given as \( n_{p2}(s) = -k(-c_1 s^{n-1} - \cdots - c_{n-1} s - c_n) \) and \( d_{p2}(s) = s^n + (a_1 - kc_1 a_1 + kc_2) s^{n-1} + \cdots + (a_{n-1} - kc_1 a_{n-1} + kc_n) s + (a_n - kc_1 a_n) \). To simplify, rewrite \( d_{p2}(s) \) as \( (1 - kc_1) d_p(s) + k n_{sp}(s) \). Then, \( P_2 = \{ k n_{sp}(s) \} \{ (1 - kc_1) d_p(s) + k n_{sp}(s) \} \). Consequently, we obtain
\[ P^*(s) = \frac{n_{p2}(s) - d_{p2}(s)}{(s + \mu)(\mu p_2 + d_{p2})} = \frac{sn_{p2} - d_{p2}(s)}{(s + \mu)(\mu d_p(s) + (1 - kc_1)d_p(s))}. \]

For the plant \( (A_p, B_p, C_p) \) not in the controllable canonical form, \( c_1 \) is replaced by \( C_p B_p \). Note that \( C_p B_p = c_1 = 0 \) when the given plant \( P \) has relative degree greater than 1.

Finally, we prove the theorem by letting \( P^*(s) = n_{p2}(s)/d_{p2}(s) \). Note that \( P^*(s) \) is strongly stabilizable if and only if \( d_{p2}(s) \) has the same sign at all the nonnegative real zeros of \( P^*(s) \) and that the zeros of \( P^*(s) \) consist of the roots of \( d_{p2}(s) \) with a zero at \( s = +\infty \) added. Let \( \sigma_i \) denote any nonnegative real root of \( d_p(s) \). At \( s = \sigma_i \geq 0 \), \( d_{p2}(\sigma_i) = (\sigma_i + \mu) [k(\sigma_i + \mu) n_{p2}(\sigma_i)] \), where \( \sigma_i + \mu > 0 \). At \( s = +\infty \), \( d_{p2}(+\infty) > 0 \) since the coefficient of the highest-order term \( s^{n+1} \) is 1. Therefore, \( P^*(s) \) is strongly stabilizable if and only if all the \( n_{p2}(\sigma_i) \)'s have the positive signs. This implies that if all the \( n_{p2}(\sigma_i) \)'s have the same signs then \( P^*(s) \) is strongly stabilizable with appropriate \( k \) guaranteeing \( k \neq 0 \) and \( \bar{C}_p B_p \). This completes the proof. ■

Note that in the proof of Theorem 2, the sign of \( d_{p2}(\sigma_i) \) is not affected by \( \mu \) and hence a PFC exists irrespective of the choice of \( \mu \) whenever the conditions of Theorem 2 are met. Note also that Theorem 2 is also necessary for the PFC to exist when the given plant \( P \) has relative degree greater than 1.

The following corollary provides two easy cases regarding the existence of a PFC \( V \).

Corollary 1: If one of the following holds, then a stable PFC \( V \) exists.
(a) On \( \mathbb{R}_+ \), \( d_p(s) \) does not have any zeros.
(b) On \( \mathbb{R}_+ \), \( d_p(s) \) has a unique zero.

Corollary 1 covers a large class of systems since it is related with the poles of \( P(s) \) on \( \mathbb{R}_+ \), only, which means that the poles outside \( \mathbb{R}_+ \) are not of interest.

We now provide a design algorithm for the PFC \( V \) for a given plant \( P \). The key in the algorithm is to design a simultaneous stabilizer \( C_{ss}(s) \) for \( P_1(s) \) and \( P_2(s) \). From the proof of Lemma 1 in [3], it is seen that \( C_{ss}(s) \) can be designed as \( C_{ss}(s) = (X_1 + M_1 C)/Y_1 - N_1 C \), where \( C_s(s) \) is a strong stabilizer for \( P^*(s) \). Since in our case \( N_1 = 1/(s + \mu), M_1 = s/(s + \mu), X_1 = \mu, \) and \( Y_1 = 1 \), the simultaneous stabilizer is given by
\[ C_{ss}(s) = \left( \mu + \frac{s}{s + \mu} C \right)/ \left( 1 - \frac{1}{s + \mu} C \right). \]

The procedure to design \( C_{ss}(s) \) for \( P^*(s) \) is omitted here, but can be found in [3], [10].

Algorithm 1: For a given plant \( P \), let \( P(s) = C_p(s I - A_p)^{-1} B_p = n_p(s)/d_p(s) \).

V1 If the condition in Theorem 2 is satisfied, compute \( P^*(s) \) with some \( k \neq 0, \neq 1/(C_p B_p) \) and \( \mu > 0 \).

V2 Find a strong stabilizer \( C_s(s) \) for \( P^*(s) \) [3], [10].

V3 The simultaneous stabilizer for \( P_1(s) \) and \( P_2(s) \) is given by
\[ C_{ss}(s) = \frac{\mu + (s/(s + \mu)) C}{1 - (1/(s + \mu)) C} \]

V4 Realize the controller \( C_{ss}(s) \) as
\[ \dot{\xi} = A_{10} \xi - A_{11} \xi, \quad \ddot{u} = A_{10} \eta + a_{11} \xi + b_u u, \quad (11) \]

Note that the minus signs appear in the realization due to the convention of the negative feedback in frequency domain.

V5 Finally, a PFC \( V \) is given as
\[ \dot{\eta} = A_{00} \eta + A_{01} y_v, \quad \dot{y}_v = A_{10} \eta + a_{11} \xi + b_u u, \quad (12) \]

in which \( b_u = (1/k) - C_p B_p \).

4. AN EXAMPLE

Consider the plant \( P \) given as
\[ \begin{cases} x = \begin{bmatrix} 5 & -4 & -8 \\ 1 & -1 & -1 \\ 3 & -2 & -5 \end{bmatrix} x + \begin{bmatrix} 1 \end{bmatrix} u, \\ y_p = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x. \end{cases} \]

Note that the transfer function of the plant is \( P(s) = (s - 1)/(s^3 + s^2 + s + 1) \) and thus, the plant has relative degree 2 and is of non-minimum phase.
For the plant $P$, a stable PFC $V$ exists from Corollary 1 because the poles of $P(s)$ are at $s = -1, \pm j$. Letting $k = 1$ and $\mu = 1$, $P^*(s)$ is computed as $P^*(s) = -(s^2 + 1)/(s^3 + 2s^2 + s)$ and a strong stabilizer for $P^*(s)$ is designed as $C_u(s) = -(s^2 + 2s + 1)/(s^2 + 2s + 3)$. Then, a simultaneous stabilizer for $P_1(s)$ and $P_2(s)$ is derived as $C_{ss}(s) = (s + 3)/(s^2 + 3s + 4)$ and one of its realization is

$$
\dot{\zeta} = \begin{bmatrix} -3 & -4 \\ 1 & 0 \end{bmatrix} \zeta + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bar{y}, \quad \bar{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \zeta.
$$

Therefore a stable PFC $V$ is designed as

$$
\dot{\eta} = \begin{bmatrix} -3 & -4 \\ 1 & 0 \end{bmatrix} \eta + \begin{bmatrix} -1 \\ 0 \end{bmatrix} y_v,
$$

$\dot{y}_v = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \eta + u,$

since $k = 1$ and $b_v = (1/k) - C_vB_p = 1$.

One can check that the designed PFC $V$ indeed solves the problem by examining the parallel interconnected system $P + V$

$$
\dot{z} = \begin{bmatrix} 5 & -4 & -8 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 3 & -2 & -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & -4 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u,
$$

$y = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} z$.

5. CONCLUSIONS

We have considered the design problem of a stable PFC which makes the augmented system with PFC be of minimum phase and have relative degree one. Based on the simultaneous stabilization problem, we also derived an existence condition of a stable PFC and proposed an algorithm for designing it.

Further research interests include the extension of the result to the multi-input multi-output system, as well as considering a nonlinear version of the problem.

APPENDIX

**Lemma 3:** Suppose that $b_v \neq 0$ and $C_vB_p + b_v \neq 0$. Then the triplet $(A, B, C)$ in (7) is controllable and observable.

**Proof:** (Controllability) The proof is done by the PBH rank test. For all $\lambda \in \mathbb{C}$, it holds that

$$
\text{rank} \begin{bmatrix} \lambda I & B \end{bmatrix} = \text{rank} \begin{bmatrix} A_p - kB_pC_pA_p - \lambda I & -kB_p \\ \{ A_p - kB_pC_pA_p - \lambda I & -kB_p \} \times \begin{bmatrix} I \\ -C_pA_p - 1/k \end{bmatrix} \end{bmatrix} = \text{rank} \begin{bmatrix} A_p - \lambda I & B_p \end{bmatrix} = n,
$$

where we used the fact that $\text{rank}(AB) = \text{rank}(A)$ for any nonsingular matrix $B$. The last equality comes from the controllability of the pair $(A_p, B_p)$.

(Observability) Let the observability matrix of $(\tilde{A}, \tilde{C})$ be $\tilde{O} = [\tilde{C}; \tilde{C}A; \cdots ; \tilde{C}A^{n-1}]$. Then, each row vector $\tilde{C}A^j$ is obtained as below.

$$
\tilde{C}A^j = \begin{bmatrix} * & \cdots & -kb_v \\ * & \cdots & -kb_v \\ \vdots & \ddots & \vdots \\ * & \cdots & -kb_v \\ * & \cdots & -kb_v \end{bmatrix} [C_p; C_pA_p; \cdots ; C_pA_p^{n-1}],
$$

where $*$ means the element which is not important, and we used the fact that $kC_pB_p + kb_v = 1$. Therefore, $\tilde{O}$ is represented as

$$
\tilde{O} = \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} C_p \\ C_pA_p \end{bmatrix} & \begin{bmatrix} C_pA_p^{n-2} & C_pA_p^{n-1} \end{bmatrix} & \cdots & \begin{bmatrix} C_pA_p^{n-2} & C_pA_p^{n-1} \end{bmatrix} \end{bmatrix}. \end{bmatrix}
$$

Since the pair $(C_p, A_p)$ is observable and $\text{rank}(BA) = \text{rank}(A)$ for any nonsingular matrix $B$, $\text{rank}(\tilde{O}) = n$. This completes the proof.

**REFERENCES**


